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Solvability and Convergence Analysis of The Multi-Step Numerical Method for Third-Kind Volterra Integral equations involving a vanishing delay

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Abstract

We investigate the application of collocation schemes to a class of Volterra delay integral equations that feature compact and noncompact operators. This class of integral equations with vanishing delays is solved numerically using multi-step collocation techniques implemented on various mesh types. The study includes an analysis of the associated operators and establishes the existence, uniqueness, and regularity of the exact solution. We prove the existence and uniqueness of the collocation solutions on different meshes. Additionally, the convergence criteria and the corresponding order of convergence are presented for each mesh type. The validity of the convergence analysis is verified through numerical examples.

Keywords: third-kind Volterra integral equations; vanishing delays; multi-step collocation scheme; compact and non compact operators.

1 Introduction

We present an analytical and numerical study of third-kind Volterra delay integral equations (VDIEs) with proportional delays

$$x^{\lambda}y(x) = \quad f(x) + (\mathcal{K}_{\alpha}y)(x) + (\mathcal{K}_{\alpha,q}y)(x), \qquad x \in I = [0, X], \tag{1}$$

with the Volterra integral operator $\mathcal{K}_{\alpha}$ given by the following expression:

$$(\mathcal{K}_{\alpha}y)(\nu) = \int_0^{\nu} (\nu - w)^{-\alpha} K(\nu, w) y(w) \mathrm{d}w,$$
$$(\mathcal{K}_{\alpha,q}y)(\nu) = \int_0^{q\nu} (\nu - w)^{-\alpha} K_D(\nu, w) y(w) \mathrm{d}w, \quad q \in (0, 1),$$

and $\alpha \in [0, 1)$, $\lambda > 0$, $f(x) = x^{\lambda}h(x)$ with some continuous function h. Also,

$$K(\nu, w) \in C(\Gamma), K_D(\nu, w) \in C(\Gamma_D)$$

with $\Gamma := \{(\nu, w) : 0 \le w \le \nu \le X\}$, $\Gamma_D := \{(\nu, w) : 0 \le w \le q\nu \le X\}$, and for $\alpha + \lambda \ge 1$, have the form

$$K(\nu, w) = w^{\alpha+\lambda-1}\hat{K}(\nu, w), \quad K_D(\nu, w) = w^{\alpha+\lambda-1}\hat{K}_D(\nu, w),$$

where $\hat{K}(\nu, w) \in C(\Gamma), \hat{K}_D(\nu, w) \in C(\Gamma_D)$.

Allaei et al. [3] have studied the following equation

$$\tau^{\lambda}y(\tau) = f(\tau) + (\mathcal{K}_{\alpha}y)(\tau), \quad \tau \in I. \tag{2}$$

The spline collocation method in piecewise polynomial spaces was applied to the third-kind Volterra integral equation (2). The compactness of the operator for the equivalent third-kind Volterra integral equation (1), under certain conditions, guarantees the unique solvability of the resulting algebraic system for all sufficiently small mesh diameters. However, for a noncompact operator, the solvability of this system is not assured on either uniform or classical graded meshes. The solvability problem was addressed through the introduction of a modified graded mesh. Using these meshes, they prove that the collocation solutions achieve the optimal order of global convergence. For the numerical solution of the third-kind VIE (2), Bhat et al. [6] developed an approach utilizing Lagrange, modified Lagrange, and barycentric Lagrange polynomial approximations, for which they also established convergence. A special category of third-kind VDIEs is defined by the presence of cordial Volterra integral operators(CVIEs), which were first defined by Vainikko (2009) [35] and [36]. Also, CVIEs has been investigated by some authors [31] and [3]. Cordial operators are also utilized in the context of Volterra integro-differential equations (VIDEs), for instance, in study by Jiang [17]. Song et al. [33] considered a spacial type of the equation (1) with $\alpha = 0, \lambda = 1$ (noncompact operators) in the nonlinear form

$$\tau y(\tau) = \tau h(\tau) + \int_0^{\tau} K(\tau, z) W_0(y(z)) dz + \int_0^{q\tau} K_D(\tau, z) W_1(y(z)) dz, \quad \tau \in I, \tag{3}$$

where $W_0,\ W_1 \in C^1(\mathbb{R})$ satisfy the Lipschitz condition. The analysis involved the application of iterated collocation methods to (3), accompanied by a discussion of the associated operator properties and the existence, uniqueness, and regularity of the exact solution. The research in [33] proves that, on a specially constructed graded mesh, the collocation solution not only exists and is unique but also achieves a global order of superconvergence Also, in [32], Song et al. studied piecewise collocation scheme for the linear form of the equation (3) with $W_0(y(z)) = W_1(y(z)) = y(z)$:

$$\tau y(\tau) = \tau h(\tau) + \int_0^{\tau} K(\tau, z)y(z)dz + \int_0^{q\tau} K_D(\tau, z)y(z)dz, \quad \tau \in I.$$

The existence of a unique solution to the collocation equations has been established, but only for two specially designed meshes: the modified graded mesh and the inverted graded mesh.

Aourir et al. [5] developed a solver for linear and nonlinear third-kind Volterra delay integral equations (3) using a moving least squares approximation enhanced by shifted Chebyshev polynomials. The independence of the suggested methodology from both a mesh and the domain geometry qualifies it as a meshless method. The solution is approximated via a collocation method grounded in moving least squares. The work describes the technique's formulation for the given equations and examines its convergence.

A wide range of scientific phenomena in biology, ecology, physics, and chemistry are modeled using Volterra integral equations. These models, which often incorporate functional terms and vanishing or non-vanishing delays, offer a robust mathematical framework for analyzing complex underlying processes. Such mathematical modeling classes are applicable to numerous areas, such as fluid dynamics, viscoelasticity of materials, population growth dynamics, heat conduction, epidemiology, controlled liquidation in obsolete production units, and renovation in economic systems, see [9]- [37], and references therein.

Exploiting non-uniform meshes is a seemingly viable strategy to attain higher-order superconvergence, most specifically in second-kind VIEs with vanishing delay. Ming et al. [22] pursued this goal by using quasi-geometric meshes to achieve superconvergence rates for the pantograph-type second-kind VIEs under investigation. The studies in [23, 18] and [21]-[30] explore high-rate numerical strategies developed for first and second-type VIEs featuring vanishing delays. The authors demonstrated the uniqueness, existence, and regularity of the analytical and numerical solutions. They also performed a convergence analysis based on novel compact operator theory. Multistep collocation techniques have proven highly efficient and effective for solving various delay equations. Studies cited in [10, 38] offer detailed evidence for their application to functional, Volterra, and Volterra integro-differential equations (FIEs, VIEs, VIFEs, VIDEs). These methods are particularly advantageous, combining straightforward implementation with high-order convergence and low computational complexity. A key early application of multistep collocation methods to second-kind VIEs was presented by Conte et al. [12].

Also, this revision integrates several key recent studies. Research on spectral and numerical methods for partial differential and integro-differential equations includes the work of Sabawi et al. [26] and [29], who analyzed the H^1 stability and convergence of Legendre spectral methods for non-local weakly singular problems. Related numerical analyses have been conducted by Kumar et al. [20] on uniformly convergent methods for singularly perturbed systems, and by Das et al. [34] and [15] on perturbation-based approaches for fractional Volterra–Fredholm integro-differential equations. In [25], authors present a spectral method based on fractional order Jacobi functions and their new constructed operational matrix of fractional integration for solving nonlinear weakly singular Volterra integral equations which include Abel equations. In [1], the Galerkin and the Petrov-Galerkin methods have been used to solve the nonlinear integral equation of the Urysohn type.

A significant body of recent work focuses on singular perturbation and layer-adapted techniques. This includes studies on systems with non-smooth data on graph networks [28], nonlinear systems with multiple layers [13], and parabolic problems with Robin conditions and time delay

[27]. Further contributions in this area feature convergence analyses on adaptive meshes for delay differential equations [34] and graded mesh refinement algorithms for time-delayed parabolic PDEs [19]. Investigations into systems of reaction–diffusion equations, particularly for Robin-type boundary value problems, are presented in [14] and [16].

In the domain of fractional calculus and applied models, recent investigations have explored Liouville-Caputo fractional-order Pennes bioheat flow equations solved via orthogonal polynomials [29] and [24]. Other studies examine fractional-order turbulent flow models with p-Laplacian operators [13] and semigroup-based frameworks for the optimal control of stochastic integro-differential equations [4] and [13].

2 Methodology

The corresponding collocation equation for equation (1) is given by

$$x^{\beta}u_h(x) = \quad f(x) + (\mathcal{K}_{\alpha}u_h)(x) + (\mathcal{K}_{\alpha,q}u_h)(x), \quad x \in \Delta_h, \tag{4}$$

and

$$\Delta_h := \{x_{n,j} = x_n + c_j h_n :\leq 1,\ 0 \leq n \leq N-1\},$$

where Δ_h the modified graded mesh that ensures the solvability of (4), given by

$$\begin{aligned} x_0 = 0, \qquad x_1 = \tfrac{T}{N}, \\ x_n = (X - x_1)(\tfrac{n-1}{N-1})^{\varsigma} + x_1, \quad \varsigma > 1,\ \ n = 1, 2, ..., N, \end{aligned} \tag{5}$$

and $\{c_i\}_{i=1}^{m}$ are collocation parameters, $\{x_{n,j}\}$ are collocation points and $h_n = x_{n+1} - x_n$ are step-size.

The approximation u_h on each subinterval $[x_n, x_{n+1}]$ is a polynomial of degree $m + r - 1$, expressible as

$$u_h(x_n + sh_n) = \sum_{k=0}^{r-1} \varphi_k(s) y_{n-k} + \sum_{j=1}^{m} \psi_j(s) U_{n,j}, \quad s \in [0,1],\ n = r, r+1, ..., N-1, \tag{6}$$

where $U_{n,j} = u_h(x_{n,j}) = u_h(x_n + c_j h_n)$, and

$$\varphi_k(s) = \prod_{\substack{i=0 \\ i \neq k}}^{r-1} \frac{s+i}{-k+i} \cdot \prod_{i=1}^{m} \frac{s - c_i}{-k - c_i}, \tag{7}$$

$$\psi_j(s) = \prod_{i=0}^{r-1} \frac{s+i}{c_j+i} \cdot \prod_{\substack{i=1 \\ i \neq j}}^{m} \frac{s - c_i}{c_j - c_i}. \tag{8}$$

For more detail see [11].

According to [8], delay terms introduce the main difficulty in the numerical analysis of VDIEs, presenting greater challenges than those encountered in VIEs without delays. The central objective is to locate the point $qx_{n,i}$. We assume it resides within a specific subinterval $[x_{q_{n,i}}, x_{q_{n,i}} + 1]$. This allows us to formulate its position as:

$$qx_{n,i} = x_{q_{n,i}} + c_{n,i} h_{q_{n,i}},$$

where $q_{n,i} \leq n$ is an integer and the local coordinate is $c_{n,i} = \frac{qx_{n,i} - x_{q_{n,i}}}{h_{q_{n,i}}}$ satisfies $0 \leq c_{n,i} \leq 1$). Consequently, the position of $qx_{n,i}$ can be categorized as follows:

I. Given the condition $qx_{n,1} > x_n$, the point is located in the current interval, hence $q_{n,1} = n$ for $i = 1, 2, \ldots, m$.

II. The inequality $qx_{n,1} \leq x_n < qx_{n,m}$, implies the existence of an index $i_1 \in \{1, \ldots, m-1\}$ which partitions the collocation points. Points up to i_1 lie in the previous interval ($q_{n,i_1} = n-1,\ i \leq i_1$) while subsequent points lie in the current one ($q_{n,i_1+1} = n,\ i > i_1$).

III. The condition $qx_{n,m} \leq x_n$, indicates that all collocation points $qx_{n,i}$ lie in the historical intervals prior to $[x_{n-1}, x_n]$. This means there exists positive integers $n_1 < n-1$ and $i_1 < m$ such that $q_{n,i_1} = n_1,\ i = 1, 2, ..., i_1$ and $q_{n,i_1+1} = n_1 + 1,\ i = i_1 + 1, ..., m$.

We now derive the discrete collocation equations for each of the three phases.

2.1 Volterra operator $(\mathcal{K}_\alpha u_h)(x_{n,i})$ and $(\mathcal{K}_{\alpha,q} u_h)(x_{n,i})$

In the linear operators $(\mathcal{K}_\alpha u_h)(x_{n,i})$ and $(\mathcal{K}_{\alpha,q} u_h)(x_{n,i})$, after some computations, we obtain

$$
\begin{aligned}
(\mathcal{K}_\alpha u_h)(x_{n,i}) &= \int_0^{x_{n,i}} (x_{n,i} - s)^{-\alpha} K(x_{n,i}, s) u_h(s) \mathrm{d}s, \\
&= \sum_{l=0}^{r-1} h_l \int_0^1 (x_{n,i} - t_l - sh_l)^{-\alpha} K(x_{n,i}, x_l + sh_l) u_h(x_l + sh_l) \mathrm{d}s \\
&\quad + \sum_{l=r}^{n-1} h_l \int_0^1 (x_{n,i} - x_l - sh_l)^{-\alpha} K(x_{n,i}, x_l + sh_l) u_h(x_l + sh_l) \mathrm{d}s \\
&\quad + h_n \int_0^{c_i} (x_{n,i} - x_n - sh_n)^{-\alpha} K(x_{n,i}, x_n + sh_n) u_h(x_n + sh_n) \mathrm{d}s.
\end{aligned}
$$

This equation, by using the equation (6) and after some computations, reduced to the following form

$$
\begin{aligned}
(\mathcal{K}_\alpha u_h)(x_{n,i}) &= \sum_{l=0}^{r-1} h_l \int_0^1 (x_{n,i} - x_l - sh_l)^{-\alpha} K(x_{n,i}, x_l + sh_l) u_h(x_l + sh_l) \mathrm{d}s \\
&\quad + \sum_{l=r}^{n-1} h_l^{1-\alpha} \sum_{k=0}^{r-1} \int_0^1 \left(\frac{x_n - x_l}{h_l} + c_i \omega_{n-1,l} - s\right)^{-\alpha} K(x_{n,i}, x_l + sh_l) \varphi_k(s) \mathrm{d}s u_{l-k} \\
&\quad + \sum_{l=r}^{n-1} h_l^{1-\alpha} \sum_{j=1}^{m} \int_0^1 \left(\frac{x_n - x_l}{h_l} + c_i \omega_{n-1,l} - s\right)^{-\alpha} K(x_{n,i}, x_l + sh_l) \psi_j(s) \mathrm{d}s U_{l,j} \\
&\quad + h_n^{1-\alpha} \sum_{k=0}^{r-1} \int_0^{c_i} (c_i - s)^{-\alpha} K(x_{n,i}, x_n + sh_n) \varphi_k(s) \mathrm{d}s u_{n-k} \\
&\quad + h_n^{1-\alpha} \sum_{j=1}^{m} \int_0^{c_i} (c_i - s)^{-\alpha} K(x_{n,i}, x_n + sh_n) \psi_j(s) \mathrm{d}s U_{n,j},
\end{aligned}
$$

where

$$\omega_{n,l}=\prod_{k=l}^{n}\omega_k,\quad \omega_n=\frac{h_{n+1}}{h_n}.$$

2.2 Volterra delays operator $(\mathcal{K}_{\alpha,q}u_h)(x_{n,i})$

2.2.1 phase I

Also, for phase I, linear delays operator $(\mathcal{K}_{\alpha,q}w_n)(x_{n,i})$, has the form

$$\begin{aligned}(\mathcal{K}^{(I)}_{\alpha,q}u_h)(x_{n,i}) &= \int_0^{qx_{n,i}}(x_{n,i}-s)^{-\alpha}K_D(x_{n,i},s)u_h(s)\mathrm{d}s,\\ &=\sum_{l=0}^{r-1}h_l\int_0^1(x_{n,i}-x_l-sh_l)^{-\alpha}K_D(x_{n,i},x_l+sh_l)u_h(x_l+sh_l)\mathrm{d}s\\ &\quad+\sum_{l=r}^{n-1}h_l\int_0^1(x_{n,i}-x_l-sh_l)^{-\alpha}K_D(x_{n,i},t_l+sh_l)u_h(x_l+sh_l)\mathrm{d}s\\ &\quad+h_n\int_0^{c_{n,i}}(x_{n,i}-x_n-sh_n)^{-\alpha}K_D(x_{n,i},x_n+sh_n)u_h(x_n+sh_n)\mathrm{d}s.\end{aligned}$$

Substituting (6) in this equation, we have

$$\begin{aligned}(\mathcal{K}^{(I)}_{\alpha,q}u_h)(x_{n,i}) &= \sum_{l=0}^{r-1}h_l\int_0^1(x_{n,i}-x_l-sh_l)^{-\alpha}K_D(x_{n,i},x_l+sh_l)u_h(x_l+sh_l)\mathrm{d}s\\ &+\sum_{l=r}^{n-1}\sum_{k=0}^{r-1}h_l^{1-\alpha}\int_0^1(\frac{x_n-x_l}{h_l}+c_i\omega_{n-1,l}-s)^{-\alpha}K_D(x_{n,i},x_l+sh_l)\varphi_k(s)y_{l-k}\mathrm{d}s\\ &+\sum_{l=r}^{n-1}\sum_{j=1}^{m}h_l^{1-\alpha}\int_0^1(\frac{x_n-x_l}{h_l}+c_i\omega_{n-1,l}-s)^{-\alpha}K_D(x_{n,i},x_l+sh_l)\psi_j(s)U_{l,j}\mathrm{d}s\\ &+h_n^{1-\alpha}\sum_{k=0}^{r-1}\int_0^{c_{n,i}}(c_i-s)^{-\alpha}K_D(x_{n,i},x_n+sh_n)\varphi_k(s)y_{n-k}\mathrm{d}s\\ &+h_n^{1-\alpha}\sum_{j=1}^{m}\int_0^{c_{n,i}}(c_i-s)^{-\alpha}K_D(x_{n,i},x_n+sh_n)\psi_j(s)U_{n,j}\mathrm{d}s.\end{aligned}$$

2.3 Phase II $(\mathcal{K}_{\alpha,q}u_h)(x_{n,i})$

2.3.1 Phase II-A

In Phase Phase II-A, let $i_1 < m$ be the index such that $q_{n,i_1} = n-1$ for $i \leq i_1$. For these indices, the collocation equation at $x_{n,i}$ is given by

$$\begin{aligned}
(\mathcal{K}_{\alpha,q}^{(II-A)}u_h)(x_{n,i}) &= \int_0^{qx_{n,i}} (x_{n,i}-s)^{-\alpha} K_D(x_{n,i},s)u_h(s)\mathrm{d}s \\
&= \sum_{l=0}^{r-1} h_l \int_0^1 (x_{n,i}-x_l-sh_l)^{-\alpha} K_D(x_{n,i},x_l+sh_l)u_h(x_l+sh_l)\mathrm{d}s \\
&\quad + \sum_{l=r}^{n-2} h_l \int_0^1 (x_{n,i}-x_l-sh_l)^{-\alpha} K_D(x_{n,i},x_l+sh_l)u_h(x_l+sh_l)\mathrm{d}s \\
&\quad + h_{n-1} \int_0^{c_{n,i}} (x_{n,i}-x_{n-1}-sh_{n-1})^{-\alpha} K_D(x_{n,i},x_{n-1}+sh_{n-1})u_h(x_{n-1}+sh_{n-1})\mathrm{d}s.
\end{aligned}$$

Now, by substituting (6) in this equation, we get

$$\begin{aligned}
(\mathcal{K}_{\alpha,q}^{(II-A)}u_h)(x_{n,i}) &= \sum_{l=0}^{r-1} h_l \int_0^1 (x_{n,i}-x_l-sh_l)^{-\alpha} K_D(x_{n,i},x_l+sh_l)u_h(x_l+sh_l)\mathrm{d}s \\
&\quad + \sum_{l=r}^{n-2}\sum_{k=0}^{r-1} h_l^{1-\alpha} \int_0^1 (\frac{x_n-x_l}{h_l}+c_i\omega_{n-1,l}-s)^{-\alpha} K_D(x_{n,i},x_l+sh_l)\varphi_k(s)\mathrm{d}s y_{l-k} \\
&\quad + \sum_{l=r}^{n-2}\sum_{j=1}^{m} h_l^{1-\alpha} \int_0^1 (\frac{x_n-x_l}{h_l}+c_i\omega_{n-1,l}-s)^{-\alpha} K_D(x_{n,i},x_l+sh_l)\psi_j(s)\mathrm{d}s U_{l,j} \\
&\quad + h_{n-1}^{1-\alpha}\sum_{k=0}^{r-1} \int_0^{c_{n,i}} (1+c_i\omega_{n-1}-s)^{-\alpha} K_D(x_{n,i},x_{n-1}+sh_{n-1})\varphi_k(s)\mathrm{d}s y_{n-k-1} \\
&\quad + h_{n-1}^{1-\alpha}\sum_{j=1}^{m} \int_0^{c_{n,i}} (1+c_i\omega_{n-1}-s)^{-\alpha} K_D(x_{n,i},x_{n-1}+sh_{n-1})\psi_j(s)\mathrm{d}s U_{n-1,j}.
\end{aligned}$$

2.3.2 Phase II-B

When $i > i_1$, the equation at at $x_{n,i}$ becomes

$$\begin{aligned}
(\mathcal{K}_{\alpha,q}^{(II-B)}u_h)(x_{n,i}) &= \int_0^{qx_{n,i}} (x_{n,i}-s)^{-\alpha} K_D(x_{n,i},s)u_h(s)\mathrm{d}s \\
&= \sum_{l=0}^{r-1} h_l \int_0^1 (x_{n,i}-x_l-sh_l)^{-\alpha} K_D(x_{n,i},x_l+sh_l)u_h(x_l+sh_l)\mathrm{d}s \\
&\quad + \sum_{l=r}^{n-1} h_l \int_0^1 (x_{n,i}-x_l-sh_l)^{-\alpha} K_D(x_{n,i},x_l+sh_l)u_h(x_l+sh_l)\mathrm{d}s \\
&\quad + h_n \int_0^{c_{n,i}} (x_{n,i}-x_n-sh_n)^{-\alpha} K_D(x_{n,i},x_n+sh_n)u_h(x_n+sh_n)\mathrm{d}s.
\end{aligned}$$

From equation (6) and this equation, we have

$$
\begin{aligned}
(\mathcal{K}_{\alpha,q}^{(II-B)}u_h)(t_{n,i}) &= \sum_{l=0}^{r-1} h_l \int_0^1 (x_{n,i} - x_l - sh_l)^{-\alpha} K_D(x_{n,i}, x_l + sh_l) u_h(x_l + sh_l)\mathrm{d}s \\
&+ \sum_{l=r}^{n-1}\sum_{k=0}^{r-1} h_l^{1-\alpha} \int_0^1 (\frac{x_n - x_l}{h_l} + c_i\omega_{n-1,l} - s)^{-\alpha} K_D(x_{n,i}, x_l + sh_l)\varphi_k(s) y_{l-k}\mathrm{d}s \\
&+ \sum_{l=r}^{n-1}\sum_{j=1}^{m} h_l^{1-\alpha} \int_0^1 (\frac{x_n - x_l}{h_l} + c_i\omega_{n-1,l} - s)^{-\alpha} K_D(x_{n,i}, x_l + sh_l)\psi_j(s) U_{l,j}\mathrm{d}s \\
&+ h_n^{1-\alpha} \sum_{k=0}^{r-1} \int_0^{c_{n,i}} (c_i - s)^{-\alpha} K_D(x_{n,i}, x_n + sh_n)\varphi_k(s) y_{n-k}\mathrm{d}s \\
&+ h_n^{1-\alpha} \sum_{j=1}^{m} \int_0^{c_{n,i}} (c_i - s)^{-\alpha} K_D(x_{n,i}, x_n + sh_n)\psi_j(s) U_{n,j}\mathrm{d}s.
\end{aligned}
$$

2.3.3 Phase III

For Phase III, we have

$$
\begin{aligned}
(\mathcal{K}_{\alpha,q}^{(III)}u_h)(x_{n,i}) &= \int_0^{qx_{n,i}} (x_{n,i} - s)^{-\alpha} K_D(x_{n,i}, s) u_h(s)\mathrm{d}s \\
&= \sum_{l=0}^{r-1} h_l \int_0^1 (x_{n,i} - x_l - sh_l)^{-\alpha} K_D(x_{n,i}, x_l + sh_l) u_h(x_l + sh_l)\mathrm{d}s \\
&+ \sum_{l=r}^{n_1-1} h_l \int_0^1 (x_{n,i} - x_l - sh_l)^{-\alpha} K_D(x_{n,i}, x_l + sh_l) u_h(x_l + sh_l)\mathrm{d}s \\
&+ h_{n_1} \int_0^{c_{n,i}} (x_{n,i} - x_{n_1} - sh_{n_1})^{-\alpha} K_D(x_{n,i}, x_{n_1} + sh_{n_1}) u_h(x_{n_1} + sh_{n_1})\mathrm{d}s.
\end{aligned}
$$

By using the equation (6), we get

$$
\begin{aligned}
(\mathcal{K}_{\alpha,q}^{(III)}u_h)(x_{n,i}) &= \sum_{l=0}^{r-1} h_l \int_0^1 (x_{n,i} - x_l - sh_l)^{-\alpha} K_D(x_{n,i}, x_l + sh_l) u_h(x_l + sh_l)\mathrm{d}s \\
&+ \sum_{l=r}^{n_1-1}\sum_{k=0}^{r-1} h_l^{1-\alpha} \int_0^1 (\frac{x_n - t_l}{h_l} + c_i\omega_{n-1,l} - s)^{-\alpha} K_D(x_{n,i}, x_l + sh_l)\varphi_k(s)\mathrm{d}s y_{l-k} \\
&+ \sum_{l=r}^{n_1-1}\sum_{j=0}^{m} h_l^{1-\alpha} \int_0^1 (\frac{x_n - x_l}{h_l} + c_i\omega_{n-1,l} - s)^{-\alpha} K_D(x_{n,i}, x_l + sh_l)\psi_j(s)\mathrm{d}s U_{l,j} \\
&+ h_{n_1}^{1-\alpha} \sum_{k=0}^{r-1} \int_0^{c_{n,i}} (\frac{x_n - x_{n_1}}{h_{n_1}} + c_i\omega_{n-1,n_1} - s)^{-\alpha} K_D(x_{n,i}, x_{n_1} + sh_{n_1})\varphi_k(s)\mathrm{d}s y_{n_1-k} \\
&+ h_{n_1}^{1-\alpha} \sum_{j=0}^{m} \int_0^{c_{n,i}} (\frac{x_n - x_{n_1}}{h_{n_1}} + c_i\omega_{n-1,n_1} - s)^{-\alpha} K_D(x_{n,i}, x_{n_1} + sh_{n_1})\psi_j(s)\mathrm{d}s U_{n_1,j}.
\end{aligned}
$$

Note that, in the general form we obtain

$$
\begin{aligned}
(\mathcal{K}_\alpha u_h)(x_{n,i}) &= \sum_{l=0}^{r-1} h_l \int_0^1 (x_{n,i} - x_l - sh_l)^{-\alpha} K(x_{n,i}, x_l + sh_l) u_h(x_l + sh_l) \mathrm{d}s \\
&+ \sum_{l=r}^{n-1} h_l^{1-\alpha} \sum_{k=0}^{r-1} \int_0^1 (\frac{x_{n,i} - x_l}{h_l} - s)^{-\alpha} K(x_{n,i}, x_l + sh_l) \varphi_k(s) \mathrm{d}s u_{l-k} \\
&+ \sum_{l=r}^{n-1} h_l^{1-\alpha} \sum_{j=1}^{m} \int_0^1 (\frac{x_{n,i} - x_l}{h_l} - s)^{-\alpha} K(x_{n,i}, x_l + sh_l) \psi_j(s) \mathrm{d}s U_{l,j} \\
&+ h_n^{1-\alpha} \sum_{k=0}^{r-1} \int_0^{c_i} (\frac{x_{n,i} - x_n}{h_n} - s)^{-\alpha} K(x_{n,i}, x_n + sh_n) \varphi_k(s) \mathrm{d}s u_{n-k} \\
&+ h_n^{1-\alpha} \sum_{j=1}^{m} \int_0^{c_i} (\frac{x_{n,i} - x_n}{h_n} - s)^{-\alpha} K(x_{n,i}, x_n + sh_n) \psi_j(s) \mathrm{d}s U_{n,j},
\end{aligned} \tag{9}
$$

and

$$(\mathcal{K}_{\alpha,q}u_h)(x_{n,i}) =$$

$$\begin{cases}
\sum_{l=0}^{r-1} h_l \int_0^1 (x_{n,i}-x_l-sh_l)^{-\alpha} K_D(x_{n,i},x_l+sh_l)u_h(x_l+sh_l)\mathrm{d}s \\
+\sum_{l=r}^{n-1} h_l^{1-\alpha} \sum_{k=0}^{r-1} \int_0^1 (\frac{x_{n,i}-x_l}{h_l}-s)^{-\alpha} K_D(x_{n,i},x_l+sh_l)\varphi_k(s)y_{l-k}\mathrm{d}s \\
+\sum_{l=r}^{n-1} h_l^{1-\alpha} \sum_{j=1}^{m} \int_0^1 (\frac{x_{n,i}-x_l}{h_l}-s)^{-\alpha} K_D(x_{n,i},x_l+sh_l)\psi_j(s)U_{l,j}\mathrm{d}s \\
+h_n^{1-\alpha} \sum_{k=0}^{r-1} \int_0^{c_{n,i}} (\frac{x_{n,i}-x_n}{h_n}-s)^{-\alpha} K_D(x_{n,i},x_n+sh_n)\varphi_k(s)y_{n-k}\mathrm{d}s \\
+h_n^{1-\alpha} \sum_{j=1}^{m} \int_0^{c_{n,i}} (\frac{x_{n,i}-x_n}{h_n}-s)^{-\alpha} K_D(x_{n,i},x_n+sh_n)\psi_j(s)U_{n,j}\mathrm{d}s, \qquad \text{Phase I}, \\
\\
\sum_{l=0}^{r-1} h_l \int_0^1 (x_{n,i}-x_l-sh_l)^{-\alpha} K_D(x_{n,i},x_l+sh_l)u_h(x_l+sh_l)\mathrm{d}s \\
+\sum_{l=r}^{n-2} h_l^{1-\alpha} \sum_{k=0}^{r-1} \int_0^1 (\frac{x_{n,i}-x_l}{h_l}-s)^{-\alpha} K_D(x_{n,i},x_l+sh_l)\varphi_k(s)\mathrm{d}s y_{l-k} \\
+\sum_{l=r}^{n-2} h_l^{1-\alpha} \sum_{j=1}^{m} \int_0^1 (\frac{x_{n,i}-x_l}{h_l}-s)^{-\alpha} K_D(x_{n,i},x_l+sh_l)\psi_j(s)\mathrm{d}s U_{l,j} \\
+h_{n-1}^{1-\alpha} \sum_{k=0}^{r-1} \int_0^{c_{n,i}} (\frac{x_{n,i}-x_{n-1}}{h_n}-s)^{-\alpha} K_D(x_{n,i},x_{n-1}+sh_{n-1})\varphi_k(s)\mathrm{d}s y_{n-k-1} \\
+h_{n-1}^{1-\alpha} \sum_{j=1}^{m} \int_0^{c_{n,i}} (\frac{x_{n,i}-x_{n-1}}{h_n}-s)^{-\alpha} K_D(x_{n,i},x_{n-1}+sh_{n-1})\psi_j(s)\mathrm{d}s U_{n-1,j}, \\
\qquad \text{Phase II-A}, \\
\\
\sum_{l=0}^{r-1} h_l \int_0^1 (x_{n,i}-x_l-sh_l)^{-\alpha} K_D(x_{n,i},x_l+sh_l)u_h(x_l+sh_l)\mathrm{d}s \\
+\sum_{l=r}^{n-1} h_l^{1-\alpha} \sum_{k=0}^{r-1} \int_0^1 (\frac{x_{n,i}-x_l}{h_l}-s)^{-\alpha} K_D(x_{n,i},x_l+sh_l)\varphi_k(s)y_{l-k} \\
+\sum_{l=r}^{n-1} h_l^{1-\alpha} \sum_{j=1}^{m} \int_0^1 (\frac{x_{n,i}-x_l}{h_l}-s)^{-\alpha} K_D(x_{n,i},x_l+sh_l)\psi_j(s)U_{l,j}\mathrm{d}s \\
+h_n^{1-\alpha} \sum_{k=0}^{r-1} \int_0^{c_{n,i}} (\frac{x_{n,i}-x_n}{h_l}-s)^{-\alpha} K_D(x_{n,i},x_n+sh_n)\varphi_k(s)y_{n-k}\mathrm{d}s \\
+h_n^{1-\alpha} \sum_{j=1}^{m} \int_0^{c_{n,i}} (\frac{x_{n,i}-x_n}{h_l}-s)^{-\alpha} K_D(x_{n,i},x_n+sh_n)\psi_j(s)U_{n,j}\mathrm{d}s, \qquad \text{Phase II-B}, \\
\\
\sum_{l=0}^{r-1} h_l \int_0^1 (x_{n,i}-x_l-sh_l)^{-\alpha} K_D(x_{n,i},x_l+sh_l)u_h(x_l+sh_l)\mathrm{d}s \\
+\sum_{l=r}^{q_{n,i}-1} h_l^{1-\alpha} \sum_{k=0}^{r-1} \int_0^1 (\frac{x_{n,i}-x_l}{h_l}-s)^{-\alpha} K_D(x_{n,i},x_l+sh_l)\varphi_k(s)\mathrm{d}s y_{l-k} \\
+\sum_{l=r}^{q_{n,i}-1} h_l^{1-\alpha} \sum_{j=0}^{m} \int_0^1 (\frac{x_{n,i}-x_l}{h_l}-s)^{-\alpha} K_D(x_{n,i},x_l+sh_l)\psi_j(s)\mathrm{d}s U_{l,j} \\
+h_{q_{n,i}}^{1-\alpha} \sum_{k=0}^{r-1} \int_0^{c_{n,i}} (\frac{x_{n,i}-x_{q_{n,i}}}{h_{q_{n,i}}}-s)^{-\alpha} K_D(x_{n,i},x_{q_{n,i}}+sh_{q_{n,i}})\varphi_k(s)\mathrm{d}s y_{q_{n,i}-k} \\
+h_{q_{n,i}}^{1-\alpha} \sum_{j=0}^{m} \int_0^{c_{n,i}} (\frac{x_{n,i}-x_{q_{n,i}}}{h_{q_{n,i}}}-s)^{-\alpha} K_D(x_{n,i},x_{q_{n,i}}+sh_{q_{n,i}})\psi_j(s)\mathrm{d}s U_{q_{n,i},j}, \quad \text{Phase III}.
\end{cases} \tag{10}$$

Now, with the definition of vectors and matrices

$$\boldsymbol{C}_l = \begin{bmatrix} \int_0^1 (x_{n,1} - x_l - sh_l)^{-\alpha} \Big(K(x_{n,1}, x_l + sh_l) + K_D(x_{n,1}, x_l + sh_l) \Big) u_h(x_l + sh_l) \mathrm{d}s \\ \vdots \\ \int_0^1 (x_{n,m} - x_l - sh_l)^{-\alpha} \Big(K(x_{n,m}, x_l + sh_l) + K_D(x_{n,m}, x_l + sh_l) \Big) u_h(x_l + sh_l) \mathrm{d}s \end{bmatrix},$$

$$A_n = \mathrm{diag}\Big(x_{n,1}, x_{n,2}, \dots, x_{n,m} \Big), \quad B_n = \Big(f(x_{n,1}), f(x_{n,2}), \dots, f(x_{n,m}) \Big)^T,$$

$$U_n = \Big(u_h(x_{n,1}), u_h(x_{n,2}), \dots, u_h(x_{n,m}) \Big)^T = \Big(U_{n,1}, U_{n,2}, \dots, U_{n,m} \Big)^T,$$

$$\Big(\mathcal{W}^{\kappa,w}_{\alpha,\lambda} \Big)_{i,r} = \Big(\int_0^{\kappa} (\frac{x_{n,i} - x_\lambda}{h_\lambda} - s)^{-\alpha} K(x_{n,i}, x_\lambda + \rho h_\lambda) w_r(\rho) \mathrm{d}\rho \Big),$$

$$\Big(\mathcal{W}^{\kappa,w}_{D,\alpha,\lambda} \Big)_{i,r} = \Big(\int_0^{\kappa} (\frac{x_{n,i} - x_\lambda}{h_\lambda} - s)^{-\alpha} K_D(x_{n,i}, x_\lambda + \rho h_\lambda) w_r(\rho) \mathrm{d}\rho \Big),$$

$$Y_l = \Big(y_l, y_{l-1}, , \dots, y_{l-r+1} \Big)^T.$$

Then, from equations (2.3.3) and (10), we get

$$\begin{cases} \Big(\boldsymbol{I} - h_n^{1-\alpha} A_n^{-\beta} \Big(\mathcal{W}^{c_i,\psi}_{\alpha,n} + \mathcal{W}^{c_{n,i},\psi}_{D,\alpha,n} \Big) \Big) \boldsymbol{U}_n = A_n^{-\beta} B_n + \sum_{l=0}^{r-1} h_l A_n^{-\beta} \boldsymbol{C}_l \\ + \sum_{l=r}^{n-1} h_l^{1-\alpha} A_n^{-\beta} \Big(\mathcal{W}^{1,\varphi}_{\alpha,l} + \mathcal{W}^{1,\varphi}_{D,\alpha,l} \Big) \boldsymbol{Y}_l + \sum_{l=r}^{n-1} h_l^{1-\alpha} A_n^{-\beta} \Big(\mathcal{W}^{1,\psi}_{\alpha,l} + \mathcal{W}^{1,\psi}_{D,\alpha,l} \Big) \boldsymbol{U}_l \\ + h_n^{1-\alpha} A_n^{-\beta} \Big(\mathcal{W}^{c_i,\varphi}_{\alpha,n} + \mathcal{W}^{c_{n,i},\varphi}_{D,\alpha,n} \Big) \boldsymbol{Y}_n, \qquad \text{Phase I}, \\ \\ \Big(\boldsymbol{I} - h_n^{1-\alpha} A_n^{-\beta} \mathcal{W}^{c_i,\psi}_{\alpha,n} \Big) \boldsymbol{U}_n = A_n^{-\beta} B_n + \sum_{l=0}^{r-1} h_l A_n^{-\beta} \boldsymbol{C}_l \\ + \sum_{l=r}^{n-2} h_l^{1-\alpha} A_n^{-\beta} \Big(\mathcal{W}^{1,\varphi}_{\alpha,l} + \mathcal{W}^{1,\varphi}_{D,\alpha,l} \Big) \boldsymbol{Y}_l + \sum_{l=r}^{n-2} h_l^{1-\alpha} A_n^{-\beta} \Big(\mathcal{W}^{1,\psi}_{\alpha,l} + \mathcal{W}^{1,\psi}_{D,\alpha,l} \Big) \boldsymbol{U}_l \\ + h_n^{1-\alpha} A_n^{-\beta} \Big(\mathcal{W}^{c_i,\varphi}_{\alpha,n} \Big) \boldsymbol{Y}_n + h_{n-1}^{1-\alpha} A_n^{-\beta} \Big(\mathcal{W}^{c_{n,i},\varphi}_{D,\alpha,n-1} + \mathcal{W}^{1,\varphi}_{\alpha,n-1} \Big) \boldsymbol{Y}_{n-1} \\ + h_{n-1}^{1-\alpha} A_n^{-\beta} \Big(\mathcal{W}^{c_{n,i},\psi}_{D,\alpha,n-1} + \mathcal{W}^{1,\psi}_{\alpha,n-1} \Big) \boldsymbol{U}_{n-1}, \quad i = 1, 2, ..., v_n, \qquad \text{Phase II-A}, \\ \\ \Big(\boldsymbol{I} - h_n^{1-\alpha} A_n^{-\beta} \Big(\mathcal{W}^{c_i,\psi}_{\alpha,n} + \mathcal{W}^{c_{n,i},\psi}_{D,\alpha,n} \Big) \Big) \boldsymbol{U}_n = A_n^{-\beta} B_n + \sum_{l=0}^{r-1} h_l A_n^{-\beta} \boldsymbol{C}_l \\ + \sum_{l=r}^{n-1} h_l^{1-\alpha} A_n^{-\beta} \Big(\mathcal{W}^{1,\varphi}_{\alpha,l} + \mathcal{W}^{1,\varphi}_{D,\alpha,l} \Big) \boldsymbol{Y}_l + \sum_{l=r}^{n-1} h_l^{1-\alpha} A_n^{-\beta} \Big(\mathcal{W}^{1,\psi}_{\alpha,l} + \mathcal{W}^{1,\psi}_{D,\alpha,l} \Big) \boldsymbol{U}_l \\ + h_n^{1-\alpha} A_n^{-\beta} \Big(\mathcal{W}^{c_i,\varphi}_{\alpha,n} + \mathcal{W}^{c_{n,i},\varphi}_{D,\alpha,n} \Big) \boldsymbol{Y}_n, \quad i = v_n + 1, ..., v_m, \qquad \text{Phase II-B}, \\ \\ \Big(\boldsymbol{I} - h_n^{1-\alpha} A_n^{-\beta} \mathcal{W}^{c_i,\psi}_{\alpha,n} \Big) \boldsymbol{U}_n = A_n^{-\beta} B_n + \sum_{l=0}^{r-1} h_l A_n^{-\beta} \boldsymbol{C}_l \sum_{l=r}^{n-1} h_l^{1-\alpha} A_n^{-\beta} \mathcal{W}^{1,\varphi}_{\alpha,l} \boldsymbol{Y}_l \\ + \sum_{l=r}^{n-1} h_l^{1-\alpha} A_n^{-\beta} \mathcal{W}^{1,\psi}_{\alpha,l} \boldsymbol{U}_l + h_n^{1-\alpha} A_n^{-\beta} \mathcal{W}^{c_i,\varphi}_{\alpha,n} \boldsymbol{Y}_n + \sum_{l=r}^{q_{n,i}-1} h_l^{1-\alpha} A_n^{-\beta} \mathcal{W}^{1,\varphi}_{D,\alpha,l} \boldsymbol{Y}_l \\ + \sum_{l=r}^{q_{n,i}-1} h_l^{1-\alpha} A_n^{-\beta} \mathcal{W}^{1,\psi}_{D,\alpha,l} \boldsymbol{U}_l + h_{q_{n,i}}^{1-\alpha} A_n^{-\beta} \mathcal{W}^{c_{n,i},\varphi}_{D,\alpha,q_{n,i}} \boldsymbol{Y}_{q_{n,i}} \\ + h_{q_{n,i}}^{1-\alpha} A_n^{-\beta} \mathcal{W}^{c_{n,i},\psi}_{D,\alpha,q_{n,i}} \boldsymbol{U}_{q_{n,i}}, \qquad \text{Phase III}, \\ \\ y_{n+1} = \sum_{k=0}^{r-1} \varphi_k(1) y_{n-k} + \sum_{j=1}^{m} \psi_j(1) U_{n,j}. \end{cases} \tag{11}$$

The general form of equations (11) in triple paths are equivalent to the following equations

$$\begin{cases} \left(\boldsymbol{I} - h_n^{1-\alpha} A_n^{-\beta}\left(\mathcal{W}_{\alpha,n}^{c_i,\psi} + \mathcal{W}_{D,\alpha,n}^{c_n,i,\psi}\right)\right)\boldsymbol{U}_n \\ = A_n^{-\beta}B_n + \sum_{l=0}^{r-1} h_l A_n^{-\beta}\boldsymbol{C}_l + \sum_{l=r}^{n-1} h_l^{1-\alpha} A_n^{-\beta}\left(\mathcal{W}_{\alpha,l}^{1,\varphi} + \mathcal{W}_{D,\alpha,l}^{1,\varphi}\right)\boldsymbol{Y}_l \\ + \sum_{l=r}^{n-1} h_l^{1-\alpha} A_n^{-\beta}\left(\mathcal{W}_{\alpha,l}^{1,\psi} + \mathcal{W}_{D,\alpha,l}^{1,\psi}\right)\boldsymbol{U}_l + h_n^{1-\alpha} A_n^{-\beta}\left(\mathcal{W}_{\alpha,n}^{c_i,\varphi} + \mathcal{W}_{D,\alpha,n}^{c_n,i,\varphi}\right)\boldsymbol{Y}_n, \qquad \text{Phase I,} \\ \\ \left(\boldsymbol{I} - h_n^{1-\alpha} A_n^{-\beta}\left(\mathcal{W}_{\alpha,n}^{c_i,\psi} + S_0\mathcal{W}_{D,\alpha,n}^{c_n,i,\psi}\right)\right)\boldsymbol{U}_n \\ = A_n^{-\beta}B_n + \sum_{l=0}^{r-1} h_l A_n^{-\beta}\boldsymbol{C}_l + \sum_{l=r}^{n-2} h_l^{1-\alpha} A_n^{-\beta}\left(\mathcal{W}_{\alpha,l}^{1,\varphi} + \mathcal{W}_{D,\alpha,l}^{1,\varphi}\right)\boldsymbol{Y}_l \\ + \sum_{l=r}^{n-2} h_l^{1-\alpha} A_n^{-\beta}\left(\mathcal{W}_{\alpha,l}^{1,\psi} + \mathcal{W}_{D,\alpha,l}^{1,\psi}\right)\boldsymbol{U}_l + h_n^{1-\alpha} A_n^{-\beta}\left(\mathcal{W}_{\alpha,n}^{c_i,\varphi} + S_0\mathcal{W}_{D,\alpha,n}^{c_n,i,\varphi}\right)\boldsymbol{Y}_n \\ + h_{n-1}^{1-\alpha} A_n^{-\beta}\left(\mathcal{W}_{\alpha,n-1}^{1,\varphi} + S_0\mathcal{W}_{D,\alpha,n-1}^{1,\varphi} + S_1\mathcal{W}_{D,\alpha,n-1}^{c_n,i,\varphi}\right)\boldsymbol{Y}_{n-1} \\ + h_{n-1}^{1-\alpha} A_n^{-\beta}\left(\mathcal{W}_{\alpha,n-1}^{1,\psi} + S_0\mathcal{W}_{D,\alpha,n-1}^{1,\psi} + S_1\mathcal{W}_{D,\alpha,n-1}^{c_n,i,\psi}\right)\boldsymbol{U}_{n-1}, \qquad \text{Phase II} \\ \\ \left(\boldsymbol{I} - h_n^{1-\alpha} A_n^{-\beta}\mathcal{W}_{\alpha,n}^{c_i,\psi}\right)\boldsymbol{U}_n \\ = A_n^{-\beta}B_n + \sum_{l=0}^{r-1} h_l A_n^{-\beta}\boldsymbol{C}_l + \sum_{l=r}^{n-1} h_l^{1-\alpha} A_n^{-\beta}\mathcal{W}_{\alpha,l}^{1,\varphi}\boldsymbol{Y}_l + \sum_{l=r}^{n-1} h_l^{1-\alpha} A_n^{-\beta}\mathcal{W}_{\alpha,l}^{1,\psi}\boldsymbol{U}_l \\ + h_n^{1-\alpha} A_n^{-\beta}\mathcal{W}_{\alpha,n}^{c_i,\varphi}\boldsymbol{Y}_n + \sum_{l=r}^{q_n-1} h_l^{1-\alpha} A_n^{-\beta}\mathcal{W}_{D,\alpha,l}^{1,\varphi}\boldsymbol{Y}_l + \sum_{l=r}^{q_n-1} h_l^{1-\alpha} A_n^{-\beta}\mathcal{W}_{D,\alpha,l}^{1,\psi}\boldsymbol{U}_l \\ + h_{q_n}^{1-\alpha} A_n^{-\beta}\left(S_0\mathcal{W}_{D,\alpha,q_n}^{1,\varphi} + S_1\mathcal{W}_{D,\alpha,q_n}^{c_n,i,\varphi}\right)\boldsymbol{Y}_{q_n} \\ + h_{q_n}^{1-\alpha} A_n^{-\beta}\left(S_0\mathcal{W}_{D,\alpha,q_n}^{1,\psi} + S_1\mathcal{W}_{D,\alpha,q_n}^{c_n,i,\psi}\right)\boldsymbol{U}_{q_n} \\ + h_{q_n+1}^{1-\alpha} A_n^{-\beta} S_0\mathcal{W}_{D,\alpha,q_n+1}^{c_n,i,\varphi}\boldsymbol{Y}_{q_n+1} + h_{q_n+1}^{1-\alpha} A_n^{-\beta} S_0\mathcal{W}_{D,\alpha,q_n+1}^{c_n,i,\psi}\boldsymbol{U}_{q_n+1}, \qquad \text{Phase III,} \\ \\ \boldsymbol{Y}_{n+1} = \Phi\boldsymbol{Y}_n + \Psi\boldsymbol{U}_n, \end{cases} \tag{12}$$

where

$$\Phi = \left[\begin{array}{c|c} \mathbf{0}_{r-1,1} & \mathbf{I}_{r-1} \\ \hline \varphi_{r-1}(1) & \varphi_{r-2}(1), \cdots, \varphi_0(1), \end{array}\right], \quad \Psi = \left[\begin{array}{ccc} & \mathbf{0}_{r-1,m} & \\ \hline \psi_1(1) & \cdots & \psi_m(1) \end{array}\right],$$

$$S_0 = \operatorname{diag}(\underbrace{0, \ldots, 0}_{\text{up to the } v_n th}, 1, \ldots, 1), \quad S_1 = \operatorname{diag}(\underbrace{1, \ldots, 1}_{\text{up to the } v_n th}, 0, \ldots, 0).$$

Now, our main goal is to analyze the solvability and uniqueness of the solution of system (12) in triple paths. For this purpose, we define the coefficient matrices of system (12) in the triple paths as follows

$$\mathcal{B}_n(q) = \begin{cases} \mathcal{B}_n^{(I)}(q) = \left(\boldsymbol{I} - h_n^{1-\alpha} A_n^{-\beta}\left(\mathcal{W}_{\alpha,n}^{c_i,\psi} + \mathcal{W}_{D,\alpha,n}^{c_n,i,\psi}\right)\right), & \text{Phase I,} \\ \\ \mathcal{B}_n^{(II)}(q) = \left(\boldsymbol{I} - h_n^{1-\alpha} A_n^{-\beta}\left(\mathcal{W}_{\alpha,n}^{c_i,\psi} + S_0\mathcal{W}_{D,\alpha,n}^{c_n,i,\psi}\right)\right), & \text{Phase II,} \\ \\ \mathcal{B}_n^{(III)}(q) = \left(\boldsymbol{I} - h_n^{1-\alpha} A_n^{-\beta}\mathcal{W}_{\alpha,n}^{c_i,\psi}\right), & \text{Phase III.} \end{cases}$$

Theorem 2.1. *Provided the functions K and K_D are continuous on their respective domains Ω and Ω_D, and the sum $\alpha+\beta$ lies strictly between 0 and 1, the following holds: For any given set of collocation parameters $0 < c_1 < \cdots < c_m \leq 1$, one can find positive constants $\tilde{h}_n^{(\eta)}$ (for $\eta = I, II, III$) and h such that the matrices $\mathcal{B}_n(q)$ are invertible. This invertibility is guaranteed as long as the step size h does not exceed the maximum of the values $h_n^{(\eta)}$, which in turn is bounded by the minimum of the $\tilde{h}_n^{(\eta)}$ values.*

Proof. For given collocation parameters $\{c_i\}_{i=1}^m$, we obtain following phases:

A1. In phase I, we obtain

$$\begin{aligned}
\| \mathcal{W}_{\alpha,n}^{c_i,\psi} \|_\infty &= \max_{i=1,\ldots,m} \sum_{j=1}^m \Big| \int_0^{c_i} \Big(\frac{x_{n,i}-x_n}{h_n} - s\Big)^{-\alpha} K(x_{n,i}, x_n + sh_n)\psi_j(s)\mathrm{d}s\Big|,\\
&\leq \frac{1}{1-\alpha} \| K \|_\infty \ \Psi_m(a) \max_i c_i^{1-\alpha},\\
&\leq \frac{c_m^{1-\alpha}}{1-\alpha} \| K \|_\infty \ \Psi_m(a),
\end{aligned}$$

where the value of the constant a depends on the chosen collocation parameters $\{c_i\}$. Here, $\Psi_m(a)$ denotes the Lebesgue constant, which is defined as

$$\Psi_m(a) = \max_{s\in[0,c_m]} \sum_{j=1}^m |\psi_j(s)|.$$

Also

$$\begin{aligned}
\| \mathcal{W}_{D,\alpha,n}^{c_{n,i},\psi} \|_\infty &= \max_{i=1,\ldots,m} \sum_{j=1}^m \Big| \int_0^{c_{n,i}} \Big(\frac{x_{n,i}-x_n}{h_n} - s\Big)^{-\gamma} K_D,(x_{n,i}, x_n + sh_n)\psi_j(s)\mathrm{d}s\Big|,\\
&\leq \frac{1}{1-\alpha} \| K_D \|_\infty \ \Phi_m(b) \max_i \{(c_i - c_{n,i})^{1-\alpha} - c_i^{1-\alpha}\},\\
&\leq \frac{(c_m - c_{n,1})^{1-\alpha} - c_1^{1-\alpha}}{1-\alpha} \| K_D \|_\infty \ \Psi_m(b),
\end{aligned}$$

where the value of the constant b depends on the chosen collocation parameters $c_{n,i}$. Here, $\Phi_m(b)$ denotes the Lebesgue constant, which is defined as

$$\Phi_m(b) = \max_{s\in[0,c_{n,m}]} \sum_{j=1}^m |\Psi_j(s)|.$$

Now, let $\Phi_m(c^*) = \max\{\Phi_m(a), \Phi_m(b)\}$. Then we have

$$\begin{aligned}
&\| h_n^{1-\alpha} A_n^{-\beta}\Big(\mathcal{W}_{\alpha,n}^{c_i,\Psi} + \mathcal{W}_{D,\alpha,n}^{c_{n,i},\Psi}\Big) \|_\infty\\
&\quad = \max |h_n^{1-\alpha} A_n^{-\beta}\Big(\mathcal{W}_{\alpha,n}^{c_i,\Psi} + \mathcal{W}_{D,\alpha,n}^{c_{n,i},\Psi}\Big)|,\\
&\quad \leq |h_n^{1-\alpha}| \ \| A_n^{-\beta} \|_\infty \| \Big(\mathcal{W}_{\alpha,n}^{c_i,\Psi} + \mathcal{W}_{D,\alpha,n}^{c_{n,i},\Psi}\Big) \|_\infty,\\
&\quad \leq h_n^{1-\alpha} \| A_n^{-\beta} \|_\infty \Bigg(\| \mathcal{W}_{\alpha,n}^{c_i,\Psi} \|_\infty + \| \mathcal{W}_{D,\alpha,n}^{c_{n,i},\Psi} \|_\infty \Bigg),\\
&\quad \leq h_n^{1-\alpha}(c_1 h_n)^{-\beta} \Bigg(\frac{c_m^{1-\alpha}}{1-\alpha} \| K \|_\infty \ \Psi_m(a) + \frac{(c_m - c_{n,1})^{1-\alpha} - c_1^{1-\alpha}}{1-\alpha} \| K_D \|_\infty \ \Psi_m(b) \Bigg),\\
&\quad \leq h_n^{1-\alpha-\beta}(c_1)^{-\beta} \Bigg(\frac{c_m^{1-\alpha}}{1-\alpha} \| K \|_\infty + \frac{(c_m - c_{n,1})^{1-\alpha} - c_1^{1-\alpha}}{1-\alpha} \| K_D \|_\infty \Bigg) \Psi_m(c^*).
\end{aligned}$$

On the other hands, a necessary and sufficient condition for the validity of the relationship

$$\| h_n^{1-\alpha} A_n^{-\beta}\Big(\mathcal{W}_{\alpha,n}^{c_i,\Psi} + \mathcal{W}_{D,\alpha,n}^{c_{n,i},\Psi}\Big) \|_\infty \leq \tfrac{1}{2},$$

is to have

$$
\begin{aligned}
& h_n^{1-\alpha-\beta} c_1^{-\beta}\left(\frac{c_m^{1-\alpha}}{1-\alpha} \parallel K \parallel_\infty + \frac{(c_m-c_{n,1})^{1-\alpha}-c_1^{1-\alpha}}{1-\alpha} \parallel K_D \parallel_\infty\right)\Psi_m(c^*) \leq \tfrac{1}{2}, \\
& \Longleftrightarrow \\
& h_n^{1-\alpha-\beta} \leq \frac{1}{2c_1^{-\beta}\left(\frac{c_m^{1-\alpha}}{1-\alpha}\|K\|_\infty + \frac{(c_m-c_{n,1})^{1-\alpha}-c_1^{1-\alpha}}{1-\alpha}\|K_D\|_\infty)\right)\Psi_m(c^*)}, \\
& \Longleftrightarrow \\
& h_n \leq \left[\frac{1}{2c_1^{-\beta}\left(\frac{c_m^{1-\alpha}}{1-\alpha}\|K\|_\infty + \frac{(c_m-c_{n,1})^{1-\alpha}-c_1^{1-\alpha}}{1-\alpha}\|K_D\|_\infty\right)\Psi_m(c^*)}\right]^{\frac{1}{1-\alpha-\beta}}, \\
& \Longleftrightarrow \\
& h_n \leq \left[2c_1^{-\beta}\left(\frac{c_m^{1-\alpha}}{1-\alpha} \parallel K \parallel_\infty + \frac{(c_m-c_{n,1})^{1-\alpha}-c_1^{1-\alpha}}{1-\alpha} \parallel K_D \parallel_\infty\right)\Psi_m(c^*)\right]^{-\frac{1}{1-\alpha-\beta}}, \\
& \Longleftrightarrow \\
& h_n \leq \left[\frac{2c_1^{-\beta}}{1-\alpha}\left(c_m^{1-\alpha} \parallel K \parallel_\infty + \left((c_m-c_{n,1})^{1-\alpha}-c_1^{1-\alpha}\right) \parallel K_D \parallel_\infty\right)\Psi_m(c^*)\right]^{-\frac{1}{1-\alpha-\beta}}.
\end{aligned}
$$

Since $0 < \alpha+\beta < 1$, and $c_1 > 0$, the constant

$$
\tilde{h}_n^{(I)} = \left[\frac{2c_1^{-\beta}}{1-\alpha}\left(c_m^{1-\alpha} \parallel K \parallel_\infty + \left((c_m-c_{n,1})^{1-\alpha}-c_1^{1-\alpha}\right) \parallel K_D \parallel_\infty\right)\Psi_m(c^*)\right]^{-\frac{1}{1-\alpha-\beta}},
$$

is well-defined. Moreover, the matrix A_n is invertible and its norm is bounded by $\|A_n^{-\beta}\|_\infty \leq (c_1h_n)^{-\beta}$. Given the assumption $\nu+\gamma<1$, if the step size satisfies $h_n \leq \tilde{h}^{(I)}$, then

$$
\parallel h_n^{1-\alpha} A_n^{-\beta}\left(\mathcal{W}_{\alpha,n}^{c_i,\Psi} + \mathcal{W}_{D,\alpha,n}^{c_{n,i},\Psi}\right) \parallel_\infty \leq \tfrac{1}{2}.
$$

A2. In phase II, we have

$$
\parallel S_0 \parallel_\infty = 1,
$$

and

$$
\begin{aligned}
\parallel S_0\mathcal{W}_{D,\alpha,n}^{c_{n,i},\psi} \parallel_\infty \leq \parallel \mathcal{S}_0 \parallel_\infty \parallel \mathcal{W}_{D,\alpha,n}^{c_{n,i},\psi} \parallel_\infty
&\leq \frac{(c_m-c_{n,1})^{1-\alpha}-c_1^{1-\alpha}}{1-\alpha} \parallel K_D \parallel_\infty \Psi_m(b) \qquad , \\
&\leq \frac{(c_m-c_{n,1})^{1-\alpha}-c_1^{1-\alpha}}{1-\alpha} \parallel K_D \parallel_\infty \Psi_m(c^*).
\end{aligned}
$$

Given that $\alpha+\beta \in (0,1)$ and $c_1 > 0$, we can define the constant $\tilde{h}_n^{(II)} = \tilde{h}^{(I)}$. This definition, together with the condition $\alpha+\beta<1$, implies that for any step size $h_n \leq \tilde{h}^{(II)}$,

$$
\parallel h_n^{1-\alpha} A_n^{-\beta}\left(\mathcal{W}_{\alpha,n}^{c_i,\psi} + S_0\mathcal{W}_{D,\alpha,n}^{c_{n,i},\psi}\right) \parallel_\infty \leq \frac{1}{2}.
$$

A3. In phase III, in the same manner, we have

$$
\begin{aligned}
\parallel h_n^{1-\alpha} A_n^{-\beta}\mathcal{W}_{\alpha,n}^{c_i,\psi} \parallel_\infty = &|h_n^{1-\alpha}| \parallel A_n^{-\beta} \parallel_\infty \parallel \mathcal{W}_{\alpha,n}^{c_i,\psi} \parallel_\infty, \\
&\leq h_n^{1-\alpha}(c_1h_n)^{-\beta}\frac{c_m^{1-\alpha}}{1-\alpha} \parallel K \parallel_\infty \Psi_m(a), \\
&\leq h_n^{1-\alpha-\beta}c_1^{-\beta}\frac{c_m^{1-\alpha}}{1-\alpha} \parallel K \parallel_\infty \Psi_m(c^*).
\end{aligned}
$$

On the other hands, a necessary and sufficient condition for the validity of the relationship

$$
\parallel h_n^{1-\alpha} A_n^{-\beta}\mathcal{W}_{\alpha,n}^{c_i,\psi} \parallel_\infty \leq \tfrac{1}{2},
$$

is to have

$$h_n^{1-\alpha-\beta} c_1^{-\beta} \frac{c_m^{1-\alpha}}{1-\alpha} \parallel K \parallel_\infty \Psi_m(c^*) \leq \frac{1}{2}$$

$$\Longleftrightarrow h_n^{1-\alpha-\beta} \leq \frac{1}{2c_1^{-\beta} \frac{c_m^{1-\alpha}}{1-\alpha} \|K\|_\infty \Psi_m(c^*)}$$

$$\Longleftrightarrow h_n \leq \left[\frac{1}{2c_1^{-\beta} \frac{c_m^{1-\alpha}}{1-\alpha} \|K\|_\infty \Psi_m(c^*)}\right]^{\frac{1}{1-\alpha-\beta}}$$

$$\Longleftrightarrow h_n \leq \left[\frac{2c_1^{-\beta} c_m^{1-\alpha}}{1-\alpha} \parallel K \parallel_\infty \Psi_m(c^*)\right]^{-\frac{1}{1-\alpha-\beta}}$$

Now, we set

$$\tilde{h}_n^{(III)} = \left[\frac{2c_1^{-\beta} c_m^{1-\alpha}}{1-\alpha} \parallel K \parallel_\infty \Psi_m(c^*)\right]^{-\frac{1}{1-\alpha-\beta}}. \tag{13}$$

Then we get

$$\parallel h_n^{1-\alpha} A_n^{-\beta} \mathcal{W}_{\alpha,n}^{c_i,\psi} \parallel_\infty \leq \frac{1}{2},$$

where

$$h_n \leq \tilde{h}_n^{(III)} = \left(\frac{2c_1^{-\beta} c_m^{1-\alpha}}{1-\alpha} \parallel K \parallel_\infty \Psi_m(c^*)\right)^{-\frac{1}{1-\alpha-\beta}}.$$

Now, acording the cases A1-A3, $\mathcal{B}_n(q)$ for $n = 0, 1, \ldots, N-1$, are invertible. Therefore, the system of equations (12) has a unique solution and the proof is complete. □

Algorithm
1. Input the values: X, N, r, q, m, ς, and c_i for every i from 1 to m
2. Provide $y_1, \ldots, y_r$ and $u_h(x_l + sh_l), l = 0, \ldots, r-1$ as the starting values.
3. Using Equation (7), (8) calculate the values for $\varphi_k(s)$ and $\psi_j(s)$.
4. For $i = 1, \cdots, m$, put $q_{0,i} = 0$, and $c_{0,i} = \frac{qx_{0,i} - x_{q_{0,i}}}{h_{q_{0,i}}}$.
5. For $n = 1, \ldots, N-1$,
For $i = 1, \ldots, m$,
For $k = 1, \ldots, n$,
If $x_{k-1} \leq qx_{n,i} < x_k$, then $q_{n,i} = k-1$,
If $qx_{n,i} \geq x_k$, then $q_{n,i} = n$,
Compute $c_{n,i} = \frac{qx_{n,i} - x_{q_{n,i}}}{h_{q_{n,i}}}$.
5. For $r \leq n \leq N-1$, solve the system of equations given by (11).
6. For the solution obtained in the previous step, take the values $U_{n,j}$ (for all $j = 1, \ldots, m$) and insert them into Equation (6).
7. For $r \leq n \leq N-1$, calculate the value $u_h(x_n + sh_n)$ using the results from the previous steps.

3 Convergence

Volterra integral equations with weakly singular kernels (of algebraic or logarithmic type) typically have solutions whose derivatives are unbounded at the left endpoint of the interval of integration.Note that the regularity of the unique solution for the third-kind Volterra integral equations in the following form is different and is closely connected with cordial VIEs

$$x^\lambda y(x) = f(x) + (\mathcal{K}_\alpha y)(x), x \in I = [0, X], \tag{14}$$

where

$$(\mathcal{K}_\alpha y)(x) = \int_0^x (x-s)^{-\alpha} K(x,s) y(s) \mathrm{d}s,$$

and $\alpha \in [0, 1)$, $\lambda > 0$, $f(x) = x^\lambda h(x)$ with $h \in C(I)$. Also, $K(x,s) \in C(\Gamma)$ and for $\alpha + \lambda \geq 1$, have the form

$$K(x,s) = s^{\alpha+\lambda-1} \hat{K}(x,s),$$

where $\hat{K}(x,s) \in C(\Gamma)$. From [2], the following Theorem shows the regularity of the solution $y(x)$ for VIE (14).

Theorem 3.1. *(See [2]) Let $0 < \alpha + \lambda \leq 1$, and assume that $h \in C^p(I)$ and $K \in C^p(\Gamma)$. Then the unique solution of the third-kind Volterra integral equations (14) is in $C^p(I)$. Also, for $\alpha + \lambda > 1$, if $h \in C^p(I)$ and $\hat{K} \in C^p(\Gamma)$, then $y \in C^p(I)$.*

Here, we consider third-kind Volterra integral equations with vanishing delays. As you know delay differential equations and Volterra functional equations with smooth data and proportional delays that vanish at the left endpoint of their interval of integration $I = [0, X]$ possess smooth solutions on I [7]. Like as (14), the functional equations considered in this paper, have the common feature that smooth data led to smooth solutions.

We now analyze the convergence of the multi-step collocation method for equation (1). To this end, we require the following lemma:

Lemma 3.1. *[8] Let the linear 3-KVIEs*

$$x^\beta y(x) = f(x) + \int_0^x (x-s)^{-\alpha} K(x,s) y(s) \mathrm{d}s + \int_0^{qx} (x-s)^{-\alpha} K_D(x,s) y(s) \mathrm{d}s,\ x \in I, \tag{15}$$

possesses an exact solution $y(x)$. This solution $y(x)$ obeys the following relation for any choice of collocation parameters $0 < c_1 < \cdots < c_m \leq 1$:

$$y(x_n + sh_n) = \sum_{k=0}^{r-1} \varphi_k(s) y(x_{n-k}) + \sum_{j=1}^{m} \psi_j(s) y(x_{n,j}) + h_n^{m+r} R_{m,r,n}(s),$$

where

$$R_{m,r,n}(s) = \int_{-r+1}^{1} K_{m,r}(s,\tau) y^{(m+r)}(x_n + \tau h_n) d\tau,$$

$$\begin{aligned} K_{m,r}(s,\tau) = &\frac{1}{(m+r-1)!}\Bigg((s-\tau)_+^{m+r-1} - \sum_{k=0}^{r-1} \varphi_k(s)(-k-\tau)_+^{m+r-1} \\ &- \sum_{j=1}^{m} \psi_j(s)(c_j - \tau)_+^{2m+r-1} \Bigg), \end{aligned}$$

$$(s-\tau)_+ = \begin{cases} s-\tau, & s-\tau \geq 0, \\ 0, & s-\tau < 0. \end{cases}$$

The error equation is obtained by subtracting (4) from (15):

$$x^\beta \mathcal{E}(x) = \int_0^x (x-s)^{-\alpha} K(x,s) \mathcal{E}(s) \mathrm{d}s + \int_0^{qx} (x-s)^{-\alpha} K_D(x,s) \mathcal{E}(s) \mathrm{d}s, \quad x \in I, \tag{16}$$

where the exact solution $\mathcal{E}(t) = y(x) - u_h(t)$, satisfies in

$$\mathcal{E}(x_n + sh_n) = \sum_{k=0}^{r-1} \varphi_k(s) \mathcal{E}_{n-k} + \sum_{j=1}^{m} \psi_j(s) \mathcal{E}_{n,j} + h_n^{m+r} R_{m,r,n}(s), \tag{17}$$

with $\mathcal{E}_{n-k} = \mathcal{E}(x_{n-k}),\ \mathcal{E}_{n,j} = \mathcal{E}(x_{n,j})$.

Theorem 3.2. *Assume that*

A. *The given functions in 3-KVIEs (15) satisfy $K \in C^p(\Omega)$ and $f \in C^p(I)$, with $p = m + r$.*
B. *The error of the exact multistep collocation method is defined as $\mathcal{E}(t)$.*
C. *The starting error satisfies*

$$\mathcal{E}(t_l + sh_l) = h_l^p q_l(s), \quad l = 0, 1, \ldots, r-1, \quad ||q_l(s)||_\infty \leq C_l, \tag{18}$$

where C_l is a constant.

D. *$\rho(\Phi) < 1$, where $\rho(\Phi)$, the spectral radius of the matrix Φ.*
Note that, this condition, validates the convergence of successive approximations in the iterative scheme, and ensures the contraction property needed for the existence and uniqueness of solutions.

Then

$$\| \mathcal{E} \|_\infty = O(h^{p-\alpha}),$$

where $h = \max_{0 \le l \le N-1} h_l$.

Proof. In the same manner in section 2.1, for linear operators $(\mathcal{K}_\alpha \mathcal{E}_n)(x_{n,i})$, by substituting the equations (3) and (18) in equation (3) and after some computations, we obtain

$$\begin{aligned}
(\mathcal{K}_\alpha \mathcal{E})(x_{n,i}) = &\sum_{l=0}^{r-1} h_l^{p-\alpha} \int_0^1 (\frac{x_{n,i}-x_l}{h_l} - s)^{-\alpha} K(x_{n,i}, x_l + sh_l) q_l(s) \mathrm{d}s \\
&+ \sum_{l=r}^{n-1} h_l^{1-\alpha} \sum_{k=0}^{r-1} \int_0^1 (\frac{x_{n,i}-x_l}{h_l} - s)^{-\alpha} K(x_{n,i}, x_l + sh_l) \varphi_k(s) \mathrm{d}s \mathcal{E}_{l-k} \\
&+ \sum_{l=r}^{n-1} h_l^{1-\alpha} \sum_{j=1}^{m} \int_0^1 (\frac{x_{n,i}-x_l}{h_l} - s)^{-\alpha} K(x_{n,i}, x_l + sh_l) \psi_j(s) \mathrm{d}s \mathcal{E}_{l,j} \\
&+ \sum_{l=r}^{n-1} h_l^{p-\alpha} \int_0^1 (\frac{x_{n,i}-x_l}{h_l} - s)^{-\alpha} K(x_{n,i}, x_l + sh_l) R_{m,r,l}(s) \mathrm{d}s \\
&+ h_n^{1-\alpha} \sum_{k=0}^{r-1} \int_0^{c_i} (\frac{x_{n,i}-x_n}{h_n} - s)^{-\alpha} K(x_{n,i}, x_n + sh_n) \varphi_k(s) \mathrm{d}s \mathcal{E}_{n-k} \\
&+ h_n^{1-\alpha} \sum_{j=1}^{m} \int_0^{c_i} (\frac{x_{n,i}-t_n}{h_n} - s)^{-\alpha} K(x_{n,i}, x_n + sh_n) \psi_j(s) \mathrm{d}s \mathcal{E}_{n,j} \\
&+ h_n^{p-\alpha} \int_0^{c_i} (\frac{x_{n,i}-x_n}{h_n} - s)^{-\alpha} K(x_{n,i}, x_n + sh_n) R_{m,r,n}(s) \mathrm{d}s.
\end{aligned}$$

On the other hands, by using the matrix notations we have

$$\begin{aligned}
(\mathcal{K}_\alpha \mathcal{E})(x_{n,i}) = &\sum_{l=0}^{r-1} h_l^{p-\alpha} \mathcal{W}_{\alpha,l}^{1,q_l} + \sum_{l=r}^{n-1} h_l^{1-\alpha} \mathcal{W}_{\alpha,l}^{1,\varphi} E_{1,l} + \sum_{l=r}^{n-1} h_l^{1-\alpha} \mathcal{V}_{\alpha,n-1,l}^{1,\psi,s} E_{2,l} \\
&+ \sum_{l=r}^{n-1} h_l^{p-\alpha} \mathcal{W}_{\alpha,l}^{1,R_{m,r,l}} + h_n^{1-\alpha} \mathcal{W}_{\alpha,n}^{c_i,\varphi} E_{1,n} + h_n^{1-\alpha} \mathcal{W}_{\alpha,n}^{c_i,\psi} E_{2,n} \\
&+ h_n^{p-\alpha} \mathcal{W}_{\alpha,n}^{c_i,R_{m,r,n}}.
\end{aligned} \tag{19}$$

To calculate the delay operator $(\mathcal{K}_{\alpha,q} \mathcal{E}_n)(x_{n,i})$, we will consider the following triple paths:

3.1 Phase I

For phase I, the linear delays operator $(\mathcal{K}_{\alpha,q} \mathcal{E})(x_{n,i})$, has the form

$$\begin{aligned}
(\mathcal{K}_{\alpha,q}^{(I)} \mathcal{E})(x_{n,i}) = &\sum_{l=0}^{r-1} h_l^{p-\alpha} \mathcal{W}_{\alpha,l}^{1,q_l} + \sum_{l=r}^{n-1} h_l^{1-\alpha} \mathcal{W}_{\alpha,l}^{1,\varphi} E_{1,l} + \sum_{l=r}^{n-1} h_l^{1-\alpha} \mathcal{W}_{\alpha,l}^{1,\psi} E_{2,l} \\
&+ \sum_{l=r}^{n-1} h_l^{p-\alpha} \mathcal{W}_{\alpha,l}^{1,R_{m,r,l}} + h_n^{1-\alpha} \mathcal{W}_{\alpha,n}^{c_i,\varphi} E_{1,n} + h_n^{1-\alpha} \mathcal{W}_{\alpha,n}^{c_i,\psi} E_{2,n} \\
&+ h_n^{p-\alpha} \mathcal{W}_{\alpha,n}^{c_i,R_{m,r,n}}.
\end{aligned} \tag{20}$$

Note that, equation (15) in path I is as follows

$$x_{n,i}^\beta \mathcal{E}(x_{n,i}) = (\mathcal{K}_\alpha \mathcal{E})(x_{n,i}) + (\mathcal{K}_{\alpha,q}^{(I)} \mathcal{E})(x_{n,i}). \tag{21}$$

Now by substituting equations (19) and (20) in equation (21), we get

$$
\begin{aligned}
&\left[\boldsymbol{I} - h_n^{1-\alpha} A_n^{-\beta}\left(\mathcal{W}_{\alpha,n}^{c_i,\psi} + \mathcal{W}_{D,\alpha,n}^{c_{n,i},\psi}\right) \quad -h_n^{1-\alpha} A_n^{-\beta}\left(\mathcal{W}_{D,\alpha,n}^{c_{n,i},\varphi} + \mathcal{W}_{\alpha,n}^{c_i,\varphi}\right)\right] \begin{bmatrix} E_{2,n} \\ E_{1,n} \end{bmatrix} \\
&\qquad = \sum_{l=r}^{n-1} h_l^{1-\alpha} \left[A_n^{-\beta}\left(\mathcal{W}_{\alpha,l}^{1,\psi} + \mathcal{W}_{D,\alpha,l}^{1,\psi}\right) \quad A_n^{-\beta}\left(\mathcal{W}_{\alpha,l}^{1,\varphi} + \mathcal{W}_{D,\alpha,l}^{1,\varphi}\right)\right] \begin{bmatrix} E_{2,l} \\ E_{1,l} \end{bmatrix} \\
&\qquad\quad + \sum_{l=0}^{r-1} h_l^{p-\alpha} A_n^{-\beta}\left(\mathcal{W}_{\alpha,l}^{1,q_l} + \mathcal{W}_{D,\alpha,l}^{1,q_l}\right) \\
&\qquad\quad + \sum_{l=r}^{n-1} h_l^{p-\alpha} A_n^{-\beta}\left(\mathcal{W}_{\alpha,l}^{1,R_{m,r,l}} + \mathcal{W}_{D,\alpha,l}^{1,R_{m,r,l}}\right) \\
&\qquad\quad + h_n^{p-\alpha} A_n^{-\beta}\left(\mathcal{W}_{\alpha,n}^{c_i,R_{m,r,n}} + \mathcal{W}_{D,\alpha,n}^{c_{n,i},R_{m,r,n}}\right).
\end{aligned} \tag{22}
$$

By setting $h := \max\limits_{(l)} h_l$, for $l = 0, 1, \ldots, N-1$, equation (22) simplifies to

$$
\begin{aligned}
&\left[\boldsymbol{I} - h^{1-\alpha} A_n^{-\beta}\left(\mathcal{W}_{\alpha,n}^{c_i,\psi} + \mathcal{W}_{D,\alpha,n}^{c_{n,i},\psi}\right) \quad -h^{1-\alpha} A_n^{-\beta}\left(\mathcal{W}_{D,\alpha,n}^{c_{n,i},\varphi} + \mathcal{W}_{\alpha,n}^{c_i,\varphi}\right)\right] \begin{bmatrix} E_{2,n} \\ E_{1,n} \end{bmatrix} \\
&\qquad = h^{1-\alpha} \sum_{l=r}^{n-1} \left[A_n^{-\beta}\left(\mathcal{W}_{\alpha,l}^{1,\psi} + \mathcal{W}_{D,\alpha,l}^{1,\psi}\right) \quad A_n^{-\beta}\left(\mathcal{W}_{\alpha,l}^{1,\varphi} + \mathcal{W}_{D,\alpha,l}^{1,\varphi}\right)\right] \begin{bmatrix} E_{2,l} \\ E_{1,l} \end{bmatrix} + \mathcal{O}(h^{p-\alpha}).
\end{aligned} \tag{23}
$$

Also, expression (3) with $l-1$, in place of n and $s = 1$, get

$$
E_{1,l} = \Phi E_{1,l-1} + \Psi E_{2,l-1} + h^p \tilde{\rho}_{m,r,l-1}, \quad l \geq r,
$$

or

$$
\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} E_{2,l} \\ E_{1,l} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \Psi & \Phi \end{bmatrix} \begin{bmatrix} E_{2,l-1} \\ E_{1,l-1} \end{bmatrix} + h^p \begin{bmatrix} 0 \\ \tilde{\rho}_{m,r,l-1} \end{bmatrix}, \quad l \geq r. \tag{24}
$$

Therefore, (23) and (24) can be rewritten as:

$$
\begin{aligned}
&\begin{bmatrix} \boldsymbol{I} - h^{1-\alpha} A_n^{-\beta}\left(\mathcal{W}_{\alpha,n}^{c_i,\psi} + \mathcal{W}_{D,\alpha,n}^{c_{n,i},\psi}\right) & -h^{1-\alpha} A_n^{-\beta}\left(\mathcal{W}_{D,\alpha,n}^{c_{n,i},\varphi} + \mathcal{W}_{\alpha,n}^{c_i,\varphi}\right) \\ 0 & I \end{bmatrix} \begin{bmatrix} E_{2,n} \\ E_{1,n} \end{bmatrix} \\
&\quad = h^{1-\alpha} \sum_{l=r}^{n-1} \begin{bmatrix} A_n^{-\beta}\left(\mathcal{W}_{\alpha,l}^{1,\psi} + \mathcal{W}_{D,\alpha,l}^{1,\psi}\right) & A_n^{-\beta}\left(\mathcal{W}_{\alpha,l}^{1,\varphi} + \mathcal{W}_{D,\alpha,l}^{1,\varphi}\right) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_{2,l} \\ E_{1,l} \end{bmatrix} \\
&\qquad + \begin{bmatrix} 0 & 0 \\ \Psi & \Phi \end{bmatrix} \begin{bmatrix} E_{2,n-1} \\ E_{1,n-1} \end{bmatrix} + \mathcal{O}(h^{p-\alpha}).
\end{aligned} \tag{25}
$$

3.2 Phase II

We know that phase II, includes two sub phases II-A and II-B. Now, we analyze operator $(\mathcal{K}_{\alpha,q}\mathcal{E})(x_{n,i})$, in both paths separately.

3.2.1 Phase II-A

Equation (15) in phase II-A, will be as follows

$$
x_{n,i}^{\beta} \mathcal{E}_{n,i} = (\mathcal{K}_{\alpha}\mathcal{E})(x_{n,i}) + (\mathcal{K}_{\alpha,q}^{(II-A)}\mathcal{E})(x_{n,i}). \tag{26}
$$

In this phase, there exists $i_1 < m$ such that $q_{n,i_1} = n-1$ and $q_{n,i_1+1} = n$. If $i \leq i_1$, then the linear delays operator $(\mathcal{K}_{\alpha,q}^{(II-A)}\mathcal{E})(x_{n,i})$, has the form

$$\begin{aligned}
(\mathcal{K}_{\alpha,q}^{(II-A)}\mathcal{E})(x_{n,i}) &= \int_0^{qx_{n,i}} (x_{n,i}-s)^{-\alpha} K_D(x_{n,i},s)\mathcal{E}(s)\mathrm{d}s \\
&= \sum_{l=0}^{r-1} h_l \int_0^1 (x_{n,i}-x_l-sh_l)^{-\alpha} K_D(x_{n,i},x_l+sh_l)\mathcal{E}(x_l+sh_l)\mathrm{d}s \\
&\quad + \sum_{l=r}^{n-2} h_l \int_0^1 (x_{n,i}-x_l-sh_l)^{-\alpha} K_D(x_{n,i},x_l+sh_l)\mathcal{E}(x_l+sh_l)\mathrm{d}s \\
&\quad + h_{n-1}\int_0^{c_{n,i}} (x_{n,i}-x_{n-1}-sh_{n-1})^{-\alpha} K_D(x_{n,i},x_{n-1}+sh_{n-1})\mathcal{E}(x_{n-1}+sh_{n-1})\mathrm{d}s.
\end{aligned}$$

By using the matrix symbols (2.3.3), equation (26) can be expressed in matrix form as

$$\begin{aligned}
&\begin{bmatrix} A_n - h_n^{1-\alpha}\mathcal{W}_{\alpha,n}^{c_i,\psi} & -h_n^{1-\alpha}\mathcal{W}_{\alpha,n}^{c_i,\varphi}\end{bmatrix}\begin{bmatrix}E_{2,n}\\E_{1,n}\end{bmatrix} \\
&= h_{n-1}^{1-\alpha}\begin{bmatrix}\mathcal{W}_{\alpha,n-1}^{1,\psi}+\mathcal{W}_{D,\alpha,n-1}^{c_{n,i},\psi} & \mathcal{W}_{\alpha,n-1}^{1,\varphi}+\mathcal{W}_{D,\alpha,n-1}^{c_{n,i},\varphi}\end{bmatrix}\begin{bmatrix}E_{2,n-1}\\E_{1,n-1}\end{bmatrix} \\
&+\sum_{l=r}^{n-2} h_l^{1-\alpha}\begin{bmatrix}\mathcal{W}_{\alpha,l}^{1,\psi}+\mathcal{W}_{D,\alpha,l}^{1,\psi} & \mathcal{W}_{\alpha,l}^{1,\varphi}+\mathcal{W}_{D,\alpha,l}^{1,\varphi}\end{bmatrix}\begin{bmatrix}E_{2,l}\\E_{1,l}\end{bmatrix} \\
&+\sum_{l=r}^{n-2} h_l^{p-\alpha}\left(\mathcal{W}_{D,\alpha,l}^{1,R_{m,r,l}}+\mathcal{W}_{\alpha,n-1,l}^{1,R_{m,r,l}}\right)+\sum_{l=0}^{r-1} h_l^{p-\alpha}\left(\mathcal{W}_{\alpha,l}^{1,q_l}+\mathcal{W}_{D,\alpha,l}^{1,q_l}\right) \\
&+h_{n-1}^{p-\alpha}\mathcal{W}_{\alpha,n-1}^{1,R_{m,r,n-1}}+h_n^{p-\alpha}\mathcal{W}_{\alpha,n}^{c_i,R_{m,r,n}}+h_{n-1}^{p-\alpha}\mathcal{W}_{D,\alpha,n-1}^{c_{n,i},R_{m,r,n}}.
\end{aligned} \tag{27}$$

By setting $h := \max_{(l)} h_l$, for $l = 0, 1, \ldots, N-1$, we can simplify (**??**) as

$$\begin{aligned}
&\begin{bmatrix} A_n - h^{1-\alpha}\mathcal{W}_{\alpha,n}^{c_i,\psi} & -h^{1-\alpha}\mathcal{W}_{\alpha,n}^{c_i,\varphi}\end{bmatrix}\begin{bmatrix}E_{2,n}\\E_{1,n}\end{bmatrix} \\
&= h^{1-\alpha}\begin{bmatrix}\mathcal{W}_{\alpha,n-1}^{1,\psi}+\mathcal{W}_{D,\alpha,n-1}^{c_{n,i},\psi} & \mathcal{W}_{\alpha,n-1}^{1,\varphi}+\mathcal{W}_{D,\alpha,n-1}^{c_{n,i},\varphi}\end{bmatrix}\begin{bmatrix}E_{2,n-1}\\E_{1,n-1}\end{bmatrix} \\
&+h^{1-\alpha}\sum_{l=r}^{n-2}\begin{bmatrix}\mathcal{W}_{\alpha,l}^{1,\psi}+\mathcal{W}_{D,\alpha,l}^{1,\psi} & \mathcal{W}_{\alpha,l}^{1,\varphi}+\mathcal{W}_{D,\alpha,l}^{1,\varphi}\end{bmatrix}\begin{bmatrix}E_{2,l}\\E_{1,l}\end{bmatrix}+\mathcal{O}(h^{p-\alpha}).
\end{aligned} \tag{28}$$

In the same manner in phase I, and by using the equation (24), equation (28), can be expressed in matrix form as:

$$\begin{aligned}
&\begin{bmatrix} A_n - h^{1-\alpha}\mathcal{W}_{\alpha,n}^{c_i,\psi} & -h^{1-\alpha}\mathcal{W}_{\alpha,n}^{c_i,\varphi}\\ 0 & I\end{bmatrix}\begin{bmatrix}E_{2,n}\\E_{1,n}\end{bmatrix} \\
&= h^{1-\alpha}\begin{bmatrix}\mathcal{W}_{\alpha,n-1}^{1,\psi}+\mathcal{W}_{D,\alpha,n-1}^{c_{n,i},\psi} & \mathcal{W}_{\alpha,n-1}^{1,\varphi}+\mathcal{W}_{D,\alpha,n-1}^{c_{n,i},\varphi}\\ \Psi & \Phi\end{bmatrix}\begin{bmatrix}E_{2,n-1}\\E_{1,n-1}\end{bmatrix} \\
&+h^{1-\alpha}\sum_{l=r}^{n-2}\begin{bmatrix}\mathcal{W}_{\alpha,l}^{1,\psi}+\mathcal{W}_{D,\alpha,l}^{1,\psi} & \mathcal{W}_{\alpha,l}^{1,\varphi}+\mathcal{W}_{D,\alpha,l}^{1,\varphi}\\ 0 & 0\end{bmatrix}\begin{bmatrix}E_{2,l}\\E_{1,l}\end{bmatrix}+\mathcal{O}(h^{p-\alpha}).
\end{aligned} \tag{29}$$

3.2.2 Phase II-B

For $i > i_1$, the collocation equation defined at the point $t_{n,i}$ is given by

$$x_{n,i}^{\beta}\mathcal{E}_n(x_{n,i}) = (\mathcal{K}_\alpha\mathcal{E}_n)(x_{n,i}) + (\mathcal{K}_{\alpha,q}^{(II-B)}\mathcal{E}_n)(x_{n,i}).$$

Hence in the same manner in phase II-A, we will have

$$\begin{aligned}
&\begin{bmatrix} \boldsymbol{I} - h^{1-\alpha} A_n^{-\beta}\left(\mathcal{W}_{D,\alpha,n}^{c_{n,i},\psi} + \mathcal{W}_{\alpha,n}^{c_i,\psi}\right) & -h^{1-\alpha} A_n^{-\beta}\left(\mathcal{W}_{D,\alpha,n}^{c_{n,i},\varphi} + \mathcal{W}_{\alpha,n}^{c_i,\varphi}\right) \\ 0 & I \end{bmatrix} \begin{bmatrix} E_{2,n} \\ E_{1,n} \end{bmatrix} \\
&= h^{1-\alpha} \sum_{l=r}^{n-1} \begin{bmatrix} A_n^{-\beta}\left(\mathcal{W}_{\alpha,l}^{1,\psi} + \mathcal{W}_{D,\alpha,l}^{1,\psi}\right) & A_n^{-\beta}\left(\mathcal{W}_{\alpha,l}^{1,\varphi} + \mathcal{W}_{D,\alpha,l}^{1,\varphi}\right) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_{2,l} \\ E_{1,l} \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 \\ \Psi & \Phi \end{bmatrix} \begin{bmatrix} E_{2,n-1} \\ E_{1,n-1} \end{bmatrix} + \mathcal{O}(h^{p-\alpha}).
\end{aligned} \tag{30}$$

4 Phase III

In phase III, there exist positive integers $n_1 < n$ and $i_1 < m$ for which $q_{n,i_1} = n_1 - 1$ and $q_{n,i_1+1} = n_1$. Consequently,

$$x_{n,i}^{\beta}\mathcal{E}(x_{n,i}) = (\mathcal{K}_\alpha \mathcal{E})(x_{n,i}) + (\mathcal{K}_{\alpha,q}^{(III)}\mathcal{E})(x_{n,i}), \tag{31}$$

where

$$\begin{aligned}
&(\mathcal{K}_{\alpha,q}^{(III)}\mathcal{E})(x_{n,i}) = \int_0^{qx_{n,i}} (x_{n,i} - s)^{-\alpha} K_D(x_{n,i}, s)\mathcal{E}(s)\mathrm{d}s \\
&= \sum_{l=0}^{r-1} h_l \int_0^1 (x_{n,i} - x_l - sh_l)^{-\alpha} K_D(x_{n,i}, x_l + sh_l)\mathcal{E}(x_l + sh_l)\mathrm{d}s \\
&+ \sum_{l=r}^{q_{n,i}-1} h_l \int_0^1 (x_{n,i} - x_l - sh_l)^{-\alpha} K_D(x_{n,i}, x_l + sh_l)\mathcal{E}(x_l + sh_l)\mathrm{d}s \\
&+ h_{q_{n,i}} \int_0^{c_{n,i}} (x_{n,i} - x_{q_{n,i}} - sh_{q_{n,i}})^{-\alpha} K_D(x_{n,i}, x_{q_{n,i}} + sh_{q_{n,i}})\mathcal{E}(x_{q_{n,i}} + sh_{q_{n,i}})\mathrm{d}s.
\end{aligned}$$

Therefore, equation (31) can be expressed in matrix form as:

$$\begin{aligned}
&\begin{bmatrix} \boldsymbol{I} - h^{1-\alpha} A_n^{-\beta}\mathcal{W}_{\alpha,n}^{c_i,\psi} & -h^{1-\alpha} A_n^{-\beta}\mathcal{W}_{\alpha,n}^{c_i,\varphi} \\ 0 & I \end{bmatrix} \begin{bmatrix} E_{2,n} \\ E_{1,n} \end{bmatrix} \\
&= h^{1-\alpha} \sum_{l=r}^{n-1} \begin{bmatrix} A_n^{-\beta}\mathcal{W}_{\alpha,l}^{1,\psi} & A_n^{-\beta}\mathcal{W}_{\alpha,l}^{1,\varphi} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_{2,l} \\ E_{1,l} \end{bmatrix} \\
&+ h^{1-\alpha} \begin{bmatrix} A_n^{-\beta}\mathcal{W}_{D,\alpha,q_{n,i}}^{c_{n,i},\psi} & A_n^{-\beta}\mathcal{W}_{D,\alpha,q_{n,i}}^{c_{n,i},\varphi} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_{2,q_{n,i}} \\ E_{1,q_{n,i}} \end{bmatrix} \\
&+ h^{1-\alpha} \sum_{l=r}^{q_{n,i}-1} \begin{bmatrix} A_n^{-\beta}\mathcal{W}_{D,\alpha,l}^{1,\psi} & A_n^{-\beta}\mathcal{W}_{D,\alpha,l}^{1,\varphi} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_{2,l} \\ E_{1,l} \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 \\ \Psi & \Phi \end{bmatrix} \begin{bmatrix} E_{2,n-1} \\ E_{1,n-1} \end{bmatrix} + \mathcal{O}(h^{p-\alpha}).
\end{aligned} \tag{32}$$

Now in the general form, in the 3-phases by using of the equations (25), (29), (30) and (32), we have

$$
\begin{cases}
\begin{bmatrix} \boldsymbol{I} - h^{1-\alpha} A_n^{-\beta}\left(\mathcal{W}_{\alpha,n}^{c_i,\psi} + \mathcal{W}_{D,\alpha,n}^{c_n,i,\psi}\right) & -h^{1-\alpha} A_n^{-\beta}\left(\mathcal{W}_{D,\alpha,n}^{c_n,i,\varphi} + \mathcal{W}_{\alpha,n}^{c_i,\varphi}\right) \\ 0 & I \end{bmatrix} \begin{bmatrix} E_{2,n} \\ E_{1,n} \end{bmatrix} \\
= \begin{bmatrix} A_n^{-\beta}\left(\mathcal{W}_{\alpha,n-1}^{1,\psi} + \mathcal{W}_{D,\alpha,n-1}^{1,\psi}\right) & A_n^{-\beta}\left(\mathcal{W}_{\alpha,n-1}^{1,\varphi} + \mathcal{W}_{D,\alpha,n-1}^{1,\varphi}\right) \\ \Psi & \Phi \end{bmatrix} \begin{bmatrix} E_{2,n-1} \\ E_{1,n-1} \end{bmatrix} \\
+h^{1-\alpha} \sum_{l=r}^{n-2} \begin{bmatrix} A_n^{-\beta}\left(\mathcal{W}_{\alpha,l}^{1,\psi} + \mathcal{W}_{D,\alpha,l}^{1,\psi}\right) & A_n^{-\beta}\left(\mathcal{W}_{\alpha,l}^{1,\varphi} + \mathcal{W}_{D,\alpha,l}^{1,\varphi}\right) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_{2,l} \\ E_{1,l} \end{bmatrix} \\
+\mathcal{O}(h^{p-\alpha}), \qquad \text{Phase I ,} \\
\begin{bmatrix} \boldsymbol{I} - h^{1-\alpha} A_n^{-\beta}\left(S_0\mathcal{W}_{D,\alpha,n}^{c_n,i,\psi} + \mathcal{W}_{\alpha,n}^{c_i,\psi}\right) & -h^{1-\alpha} A_n^{-\beta}\left(S_0\mathcal{W}_{D,\alpha,n}^{c_n,i,\varphi} + \mathcal{W}_{\alpha,n}^{c_i,\varphi}\right) \\ 0 & I \end{bmatrix} \begin{bmatrix} E_{2,n} \\ E_{1,n} \end{bmatrix} \\
= \begin{bmatrix} 0 & 0 \\ \Psi & \Phi \end{bmatrix} \begin{bmatrix} E_{2,n-1} \\ E_{1,n-1} \end{bmatrix} + h^{1-\alpha} \sum_{l=r}^{n-2} \begin{bmatrix} A_n^{-\beta}\left(\mathcal{W}_{\alpha,l}^{1,\psi} + \mathcal{W}_{D,\alpha,l}^{1,\psi}\right) & 0 \\ A_n^{-\beta}\left(\mathcal{W}_{\alpha,l}^{1,\varphi} + \mathcal{W}_{D,\alpha,l}^{1,\varphi}\right) & 0 \end{bmatrix}^T \begin{bmatrix} E_{2,l} \\ E_{1,l} \end{bmatrix} \\
+h^{1-\alpha} \begin{bmatrix} A_n^{-\beta}\left(\mathcal{W}_{\alpha,n-1}^{1,\psi} + S_0\mathcal{W}_{D,\alpha,n-1}^{1,\psi} + S_1\mathcal{W}_{D,\alpha,n-1}^{c_n,i,\psi}\right) & 0 \\ A_n^{-\beta}\left(\mathcal{W}_{\alpha,n-1}^{1,\varphi} + S_0\mathcal{W}_{D,\alpha,n-1}^{1,\varphi} + S_1\mathcal{W}_{D,\alpha,n-1}^{c_n,i,\varphi}\right) & 0 \end{bmatrix}^T \begin{bmatrix} E_{2,n-1} \\ E_{1,n-1} \end{bmatrix} \\
+\mathcal{O}(h^{p-\alpha}), \qquad \text{Phase II,} \\
\begin{bmatrix} \boldsymbol{I} - h^{1-\alpha} A_n^{-\beta}\mathcal{W}_{\alpha,n}^{c_i,\psi} & -h^{1-\alpha} A_n^{-\beta}\mathcal{W}_{\alpha,n}^{c_i,\varphi} \\ 0 & I \end{bmatrix} \begin{bmatrix} E_{2,n} \\ E_{1,n} \end{bmatrix} = \begin{bmatrix} A_n^{-\beta}\mathcal{W}_{\alpha,n-1}^{1,\psi} & A_n^{-\beta}\mathcal{W}_{\alpha,n-1}^{1,\varphi} \\ \Psi & \Phi \end{bmatrix} \\
\times \begin{bmatrix} E_{2,n-1} \\ E_{1,n-1} \end{bmatrix} + h^{1-\alpha} \sum_{l=r}^{n-2} \begin{bmatrix} A_n^{-\beta}\mathcal{W}_{\alpha,l}^{1,\psi} & A_n^{-\beta}\mathcal{W}_{\alpha,l}^{1,\varphi} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_{2,l} \\ E_{1,l} \end{bmatrix} \\
+h^{1-\alpha} \sum_{l=r}^{q_n-1} \begin{bmatrix} A_n^{-\beta}\mathcal{W}_{D,\alpha,l}^{1,\psi} & A_n^{-\beta}\mathcal{W}_{D,\alpha,l}^{1,\varphi} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_{2,l} \\ E_{1,l} \end{bmatrix} \\
+h^{1-\alpha} \begin{bmatrix} A_n^{-\beta} S_0\mathcal{W}_{D,\alpha,q_n+1}^{c_n,i,\psi} & A_n^{-\beta} S_0\mathcal{W}_{D,\alpha,q_n+1}^{c_n,i,\varphi} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_{2,q_n+1} \\ E_{1,q_n+1} \end{bmatrix} \\
+h^{1-\alpha} \begin{bmatrix} A_n^{-\beta}\left(S_1\mathcal{W}_{D,\alpha,q_n}^{c_n,i,\psi} + S_0\mathcal{W}_{D,\alpha,l}^{1,\psi}\right) & A_n^{-\beta}\left(S_1\mathcal{W}_{D,\alpha,q_n}^{c_n,i,\varphi} + S_0\mathcal{W}_{D,\alpha,l}^{1,\varphi}\right) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_{2,q_n} \\ E_{1,q_n} \end{bmatrix} \\
+\mathcal{O}(h^{p-\alpha}), \qquad \text{Phase III,}
\end{cases}
\tag{33}
$$

where $S_0,\ S_1$ and $A_n^{-\beta}$, are clear.

Now, we show that system (33) has a unique solution. To do this, according equations (13), and triple paths, it can be seen that the structure of the matrices of unknown coefficients are similar in both systems. Therefore, according to theorem 2.1, the coefficients matrices

$$
\mathbf{B}_n^*(q) = \begin{cases} \mathbf{B}_n^{(*,I)}(q), & \text{Phase I,} \\ \mathbf{B}_n^{(*,II)}(q), & \text{Phase II,} \\ \mathbf{B}_n^{(*,III)}(q), & \text{Phase III,} \end{cases}
$$

are bounded and have a invertible matrix. On the other hand, if we set

$$\mathbf{B}_n^{(*)}(q,h) = \left(\mathbf{B}_n^{(*)}(q) + \mathcal{O}(h)\right) = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

where

$$\mathbf{B}_n^{(*,I)}(q) = \begin{bmatrix} \boldsymbol{I} - h^{1-\alpha} A_n^{-\beta}\left(\mathcal{W}_{\alpha,n}^{c_i,\psi} + \mathcal{W}_{D,\alpha,n}^{c_{n,i},\psi}\right) & -h^{1-\alpha} A_n^{-\beta}\left(\mathcal{W}_{D,\alpha,n}^{c_{n,i},\varphi} + \mathcal{W}_{\alpha,n}^{c_i,\varphi}\right) \\ 0 & I \end{bmatrix},$$

$$\mathbf{B}_n^{(*,II)}(q) = \begin{bmatrix} \boldsymbol{I} - h^{1-\alpha} A_n^{-\beta}\left(S_0\mathcal{W}_{D,\alpha,n}^{c_{n,i},\psi} + \mathcal{W}_{\alpha,n}^{c_i,\psi}\right) & -h^{1-\alpha} A_n^{-\beta}\left(S_0\mathcal{W}_{D,\alpha,n}^{c_{n,i},\varphi} + \mathcal{W}_{\alpha,n}^{c_i,\varphi}\right) \\ 0 & I \end{bmatrix},$$

$$\mathbf{B}_n^{(*,III)}(q) = \begin{bmatrix} \boldsymbol{I} - h^{1-\alpha} A_n^{-\beta}\mathcal{W}_{\alpha,n}^{c_i,\psi} & -h^{1-\alpha} A_n^{-\beta}\mathcal{W}_{\alpha,n}^{c_i,\varphi} \\ 0 & I \end{bmatrix}.$$

Hence, in equation (33), we get

$$\begin{cases}
\begin{bmatrix} E_{2,n} \\ E_{1,n} \end{bmatrix} = \begin{bmatrix} A_n^{-\beta}\left(\mathcal{W}_{\alpha,n-1}^{1,\psi} + \mathcal{W}_{D,\alpha,n-1}^{1,\psi}\right) & A_n^{-\beta}\left(\mathcal{W}_{\alpha,n-1}^{1,\varphi} + \mathcal{W}_{D,\alpha,n-1}^{1,\varphi}\right) \\ \Psi & \Phi \end{bmatrix} \begin{bmatrix} E_{2,n-1} \\ E_{1,n-1} \end{bmatrix} \\
+h^{1-\alpha} \displaystyle\sum_{l=r}^{n-2} \begin{bmatrix} A_n^{-\beta}\left(\mathcal{W}_{\alpha,l}^{1,\psi} + \mathcal{W}_{D,\alpha,l}^{1,\psi}\right) & A_n^{-\beta}\left(\mathcal{W}_{\alpha,l}^{1,\varphi} + \mathcal{W}_{D,\alpha,l}^{1,\varphi}\right) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_{2,l} \\ E_{1,l} \end{bmatrix} \\
+\mathcal{O}(h^{p-\alpha}), \qquad \text{Phase I}, \\[2ex]
\begin{bmatrix} E_{2,n} \\ E_{1,n} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \Psi & \Phi \end{bmatrix} \begin{bmatrix} E_{2,n-1} \\ E_{1,n-1} \end{bmatrix} \\
+h^{1-\alpha} \displaystyle\sum_{l=r}^{n-2} \begin{bmatrix} A_n^{-\beta}\left(\mathcal{W}_{\alpha,l}^{1,\psi} + \mathcal{W}_{D,\alpha,l}^{1,\psi}\right) & A_n^{-\beta}\left(\mathcal{W}_{\alpha,l}^{1,\varphi} + \mathcal{W}_{D,\alpha,l}^{1,\varphi}\right) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_{2,l} \\ E_{1,l} \end{bmatrix} \\
h^{1-\alpha} \begin{bmatrix} A_n^{-\beta}\left(\mathcal{W}_{\alpha,n-1}^{1,\psi} + S_0\mathcal{W}_{D,\alpha,n-1}^{1,\psi} + S_1\mathcal{W}_{D,\alpha,n-1}^{c_{n,i},\psi}\right) & 0 \\ A_n^{-\beta}\left(\mathcal{W}_{\alpha,n-1}^{1,\varphi} + S_0\mathcal{W}_{D,\alpha,n-1}^{1,\varphi} + S_1\mathcal{W}_{D,\alpha,n-1}^{c_{n,i},\varphi}\right) & 0 \end{bmatrix}^T \begin{bmatrix} E_{2,n-1} \\ E_{1,n-1} \end{bmatrix} \\
+\mathcal{O}(h^{p-\alpha}), \qquad \text{Phase II}, \\[2ex]
\begin{bmatrix} E_{2,n} \\ E_{1,n} \end{bmatrix} = \begin{bmatrix} A_n^{-\beta}\mathcal{W}_{\alpha,n-1}^{1,\psi} & A_n^{-\beta}\mathcal{W}_{\alpha,n-1}^{1,\varphi} \\ \Psi & \Phi \end{bmatrix} \begin{bmatrix} E_{2,n-1} \\ E_{1,n-1} \end{bmatrix} \\
+h^{1-\alpha} \displaystyle\sum_{l=r}^{n-2} \begin{bmatrix} A_n^{-\beta}\mathcal{W}_{\alpha,l}^{1,\psi} & A_n^{-\beta}\mathcal{W}_{\alpha,l}^{1,\varphi} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_{2,l} \\ E_{1,l} \end{bmatrix} \\
+h^{1-\alpha} \displaystyle\sum_{l=r}^{q_n-1} \begin{bmatrix} A_n^{-\beta}\mathcal{W}_{D,\alpha,l}^{1,\psi} & A_n^{-\beta}\mathcal{W}_{D,\alpha,l}^{1,\varphi} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_{2,l} \\ E_{1,l} \end{bmatrix} \\
+h^{1-\alpha} \begin{bmatrix} A_n^{-\beta} S_0\mathcal{W}_{D,\alpha,q_n+1}^{c_{n,i},\psi} & A_n^{-\beta} S_0\mathcal{W}_{D,\alpha,q_n+1}^{c_{n,i},\varphi} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_{2,q_n+1} \\ E_{1,q_n+1} \end{bmatrix} \\
+h^{1-\alpha} \begin{bmatrix} A_n^{-\beta}\left(S_1\mathcal{W}_{D,\alpha,q_n}^{c_{n,i},\psi} + S_0\mathcal{W}_{D,\alpha,l}^{1,\psi}\right) & A_n^{-\beta}\left(S_1\mathcal{W}_{D,\alpha,q_n}^{c_{n,i},\varphi} + S_0\mathcal{W}_{D,\alpha,l}^{1,\varphi}\right) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_{2,q_n} \\ E_{1,q_n} \end{bmatrix} \\
+\mathcal{O}(h^{p-\alpha}), \qquad \text{Phase III}.
\end{cases}$$

Now, according to assumption $\rho(\Phi) < 1$, we can get a bound for

$$\begin{bmatrix} E_{2,n} \\ E_{1,n} \end{bmatrix}.$$

This establishes the desired result. $\square$

5 Numerical tests

For equation (1), we use the parameters in Table 1 and select $f(x)$ to yield the exact solution $y(x)$. The collocation parameters are given in Table 2. All numerical computations were performed using Mathematica 11 on a laptop with an Intel Core i7-1065G7 processor and 8 GB of RAM.

	λ	α	q	K	K_D	$y(x)$
Example 1, $\lambda + \alpha < 1$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$s+1$	$s+2$	$x^{\frac{7}{2}}$
Example 2, $\lambda + \alpha > 1$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{3\pi} s^{\frac{1}{3}}$	$s^{\frac{1}{3}}$	$x^{\frac{13}{4}}$
Example 3 [32], $\lambda + \alpha = 1$	1	0	$\frac{1}{2}$	s^2	$s+1$	x^5

Table 1: Assumptions of equation (1) in the given examples.

	c_1	c_2	c_3
$m = 2, c_m = 1$	0.7	1	–
$m = 2, c_m < 1$	$\frac{5}{6}$	$\frac{6}{7}$	–
$m = 3, c_m = 1$	0.6	0.7	1

Table 2: Collocation parameters.

These examples are solved using one-step [32] and multi-step collocation methods across different values of m by the modified graded mesh with $\varsigma = 2$. The numerical results, compiled in Tables 3-8, indicate that the multi-step method yields greater accuracy compared to the one-step method. For $c_m = 1$ with $r = 2, m = 2$, we consider $c_1 = 0.7$ and $c_2 = 1$ such that at these points $\rho(\Phi) = 0 < 1$. Also for the case $c_m < 1$ with $r = m = 2$, we consider $c_1 = \frac{5}{6}, c_2 = \frac{6}{7}$ such that at these points $\rho(\Phi) = 0.0836 < 1$. A comparative analysis of the numerical experiments(Tables 3-8) demonstrates that the proposed numerical method in the previous section achieves significantly higher accuracy than the one-step collocation approach developed in [32], particularly for smaller step sizes.

N	One-step $m = 2$	One-step $m = 3$	Multi-step $m = r = 2$
2^3	1.25×10^{-1}	2.14×10^{-2}	6.59×10^{-4}
2^4	5.71×10^{-2}	8.95×10^{-3}	5.12×10^{-4}
2^5	2.45×10^{-2}	8.03×10^{-3}	3.39×10^{-4}
2^6	9.92×10^{-3}	2.88×10^{-3}	8.05×10^{-5}

Table 3: Maximum errors for $c_m = 1$ in Example 1.

To assess computational efficiency, we present detailed CPU time measurements for both methods in the Tables 9-11, providing direct comparisons of algorithmic performance. In addition to its high accuracy, the multi-step method is very close to the one-step method [32] in terms of execution time.

We plot the order of convergence for the one-step $(m = 2, c_m = 1)$ and multi-step $(m = r = 2, c_m = 1)$ methods in Figures 1-3.

N	One-step $m = 2$	One-step $m = 3$	Multi-step $m = r = 2$
2^3	1.35×10^{-2}	5.83×10^{-3}	5.26×10^{-5}
2^4	7.09×10^{-3}	4.03×10^{-3}	3.80×10^{-5}
2^5	4.79×10^{-3}	3.81×10^{-3}	1.61×10^{-5}
2^6	2.37×10^{-3}	1.88×10^{-3}	5.69×10^{-6}

Table 4: Maximum errors for $c_m = 1$ in Example 2.

N	One-step $m = 2$	One-step $m = 3$	Multi-step $m = r = 2$
2^3	2.00×10^{-2}	5.79×10^{-3}	3.47×10^{-5}
2^4	5.54×10^{-3}	1.37×10^{-3}	1.67×10^{-5}
2^5	2.14×10^{-3}	9.07×10^{-4}	7.65×10^{-6}
2^6	7.00×10^{-4}	4.59×10^{-4}	2.78×10^{-6}

Table 5: Maximum errors for $c_m = 1$ in Example 3.

N	One-step	Multi-step
2^3	1.26×10^{-1}	9.41×10^{-4}
2^4	6.37×10^{-2}	7.75×10^{-4}
2^5	2.71×10^{-2}	3.70×10^{-4}
2^6	9.18×10^{-3}	1.29×10^{-4}

Table 6: Maximum errors for $c_m < 1$ with $m = r = 2$ in Example 1.

N	One-step	Multi-step
2^3	1.42×10^{-2}	1.82×10^{-4}
2^4	7.56×10^{-3}	1.34×10^{-4}
2^5	4.94×10^{-3}	5.77×10^{-5}
2^6	2.47×10^{-3}	2.06×10^{-5}

Table 7: Maximum errors for $c_m < 1$ with $m = r = 2$ in Example 2.

N	One-step	Multi-step
2^3	3.40×10^{-2}	1.61×10^{-4}
2^4	9.38×10^{-3}	6.50×10^{-4}
2^5	3.10×10^{-3}	2.53×10^{-5}
2^6	8.76×10^{-4}	8.99×10^{-6}

Table 8: Maximum errors for $c_m < 1$ with $m = r = 2$ in Example 3.

	CPU time	CPU time	CPU time
N	One-step $m = 2$	One-step $m = 3$	Multi-step $m = r = 2$
2^3	0.890	2.47	1.12
2^4	2.62	5.62	4.54
2^5	9.53	21.9	18.5
2^6	36.1	84.9	73.8

Table 9: CPU time in Example 1.

N	CPU time One-step $m = 2$	CPU time One-step $m = 3$	CPU time Multi-step $m = r = 2$
2^3	1.04	2.76	1.82
2^4	3.75	6.32	5.04
2^5	9.48	22.6	18.6
2^6	35.2	100	73.5

Table 10: CPU time in Example 2.

N	CPU time One-step $m = 2$	CPU time One-step $m = 3$	CPU time Multi-step $m = r = 2$
2^3	1.56	2.60	1.07
2^4	3.10	6.04	4.74
2^5	9.62	21.9	20.1
2^6	35.6	82.7	80.7

Table 11: CPU time in Example 3.

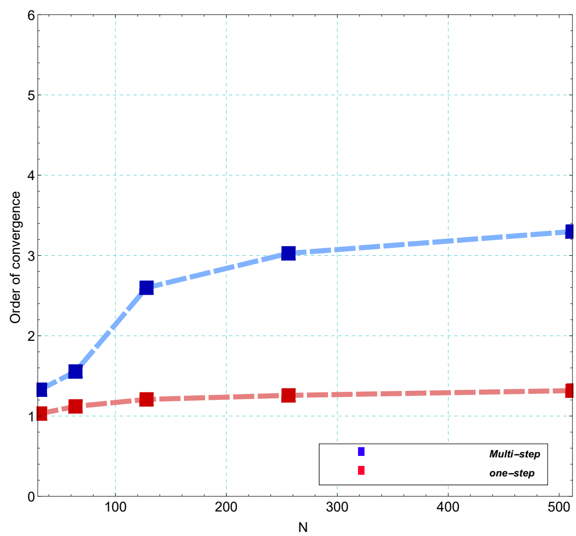

Figure 1: Plot of order of convergence for the one-step $(m = 2, c_m = 1)$ and multi-step $(m = r = 2, c_m = 1)$ scheme in Example 1.

6 Conclusion

This work has established the multi-step collocation method as a reliable and accurate numerical approach for solving a class of third-kind Volterra integral equations (VIEs) with vanishing delays. The theoretical convergence results were validated through a series of numerical examples. These tests confirmed the consistency between the numerical results and the theoretical analysis. A key direction for our future research is the analysis of multi-step collocation methods applied to challenging weakly singular VIEs with non-vanishing delays.

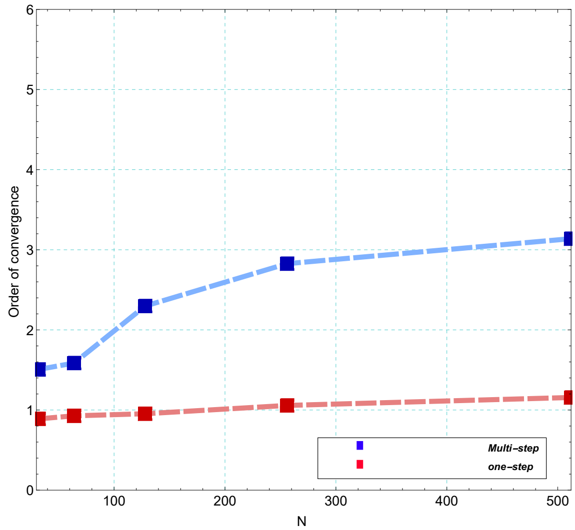

Figure 2: Plot of order of convergence for the one-step $(m = 2, c_m = 1)$ and multi-step $(m = r = 2, c_m = 1)$ scheme in Example 2.

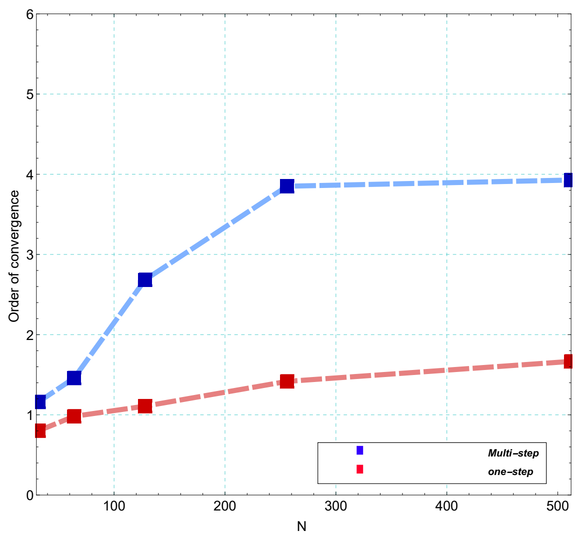

Figure 3: Plot of order of convergence for the one-step $(m = 2, c_m = 1)$ and multi-step $(m = r = 2, c_m = 1)$ scheme in Example 3.

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Conflicts of Interest Declare conflicts of interest or state "The authors declare no conflict of interest." Authors must identify and declare any personal circumstances or interest that may be perceived as inappropriately influencing the representation or interpretation of reported research results.

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